10.3 Goodness of fit for continuous distributions	231
10.4 Testing association through contingency tables	237
End-of-chapter review exercise 10	248
11 Non-parametric tests	249
11.1 Non-parametric tests	250
11.2 Single-sample sign test	251
11.3 Single-sample Wilcoxon signed-rank test	254
11.4 Paired-sample sign test	260
11.5 Wilcoxon matched-pairs signed-rank test	263
11.6 Wilcoxon rank-sum test	266
End-of-chapter review exercise 11	277
12 Probability generating functions	279
12.1 The probability generating function	280
12.2 Mean $(E(X))$ and variance $(Var(X))$ using the probability generating function	287
12.3 The sum of independent random variables	292
12.4 Three or more random variables	298
End-of-chapter review exercise 12	304
Cross-topic review exercise 2	305
Further Mechanics	
13 Projectiles	308
13.1 Motion in the vertical plane	309
13.2 The Cartesian equation of the trajectory	314
End-of-chapter review exercise 13	320
14 Equilibrium of a rigid body	321
14.1 The moment of a force	322
14.2 Centres of mass of rods and laminas	326
14.3 Centres of mass of solids	336
14.4 Objects in equilibrium	341
End-of-chapter review exercise 14	352

15 Circular motion	353
15.1 Horizontal circles	355
15.2 The 3-dimensional case	360
15.3 Vertical circles	365
End-of-chapter review exercise 15	378
16 Hooke's law	379
16.1 Hooke's law	380
16.2 Elastic potential energy	385
16.3 The work–energy principle	389
End-of-chapter review exercise 16	396
17 Linear motion under a variable force	397
17.1 Acceleration with respect to time	398
17.2 Acceleration with respect to displacement	403
End-of-chapter review exercise 17	409
18 Momentum	410
18.1 Impulse and the conservation of momentum	411
18.2 Oblique collisions and other examples	417
End-of-chapter review exercise 18	429
Cross-topic review exercise 3	430
Further Pure Mathematics 2	
19 Hyperbolic functions	431
19.1 Exponential forms of hyperbolic functions	432
19.2 Hyperbolic identities	438
19.3 Inverse hyperbolic functions	440
19.4 Logarithmic form for inverse hyperbolic functions	442
End-of-chapter review exercise 19	446
20 Matrices 2	447
20.1 Eigenvalues and eigenvectors	448
20.2 Matrix algebra	451
20.3 Diagonalisation	461
20.4 Systems of equations	465
End-of-chapter review exercise 20	471



In this chapter, you will learn how to:

- use Newton's equations of motion to determine results related to projectile motion
- use these results to solve problems involving projectiles from both ground level and raised platforms
- make use of the components of the motion to understand and solve more complicated problems.

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Mechanics, Chapter 1	Remember and use Newton's equations of motion.	1 A cyclist is travelling along a straight horizontal road at 6 m s ⁻¹ when he sees a red traffic light 50 m ahead. He stops pedalling and applies his brakes to decelerate uniformly until he stops. Find the magnitude of his deceleration.
AS & A Level Mathematics Mechanics, Chapters 1 & 3 Pure Mathematics 2 & 3, Chapter 9	Work with basic vectors, such as the components in both horizontal and vertical directions.	2 A particle is projected from the top of a cliff with initial speed 20 m s ⁻¹ vertically upwards. The particle lands at the bottom of the cliff, 50 m below the point from which it was projected. Find the time taken for the particle to reach the bottom of the cliff and the velocity of the particle at that time.

What are projectiles?

A projectile is any object that, once it has been thrown, propelled or dropped, continues to move under its own inertia and the force of gravity. We are most likely to encounter projectiles when we play sports, when we drop things, or when we observe objects after a collision. The apple that (allegedly) fell on Isaac Newton's head was a projectile.

In this chapter, you will look at the motion of **particles** that are projected at an oblique angle to the horizontal and/or vertical directions. The paths of these particles will be analysed as horizontal and vertical components. Each direction can be dealt with separately or together.

13.1 Motion in the vertical plane

In this chapter, we use the symbols a for acceleration, u for initial velocity, v for final velocity, t for time, and x for displacement. Unless stated otherwise, $g = 10 \,\mathrm{m \, s^{-2}}$.

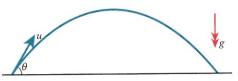
You will have met Newton's equations of motion in your earlier study of Mechanics. These equations are valid only if the acceleration of the object is constant, usually due to the mass of the object.

We use the same equations to derive some standard results for the motion of a projectile.

These equations are:

- v = u + at
- $v^2 = u^2 + 2as$
- $s = vt \frac{1}{2}at^2$

For projectile questions, we use a simplified model of reality. This means we usually assume there are no resisting forces such as air resistance. We model the projectile as a particle so we can assume there are no rotational forces (spin). We also assume that the force due to gravity is constant. The path travelled by the projectile is known as a parabolic trajectory.



Using v = u + at in the y-direction gives $v_y = u \sin \theta - gt$, and in the x-direction we can see that $v_x = u \cos \theta$. Remember that the acceleration due to gravity acts vertically downwards. Taking upwards as positive, in the vertical direction, a = -g and in the horizontal direction, a = 0.

When the particle reaches its highest point $v_y = 0$, then $u \sin \theta = gt$, or $t = \frac{u \sin \theta}{g}$. Since we can ignore air resistance, the time it takes the particle to travel from the ground to the top of its path must be the same as the time it takes to return to the ground. This means that the total flight time is $t = \frac{2u \sin \theta}{g}$.

We will focus on the distance travelled horizontally and vertically. Let x represent the horizontal displacement travelled, and y represent the vertical displacement travelled.

Then, using
$$s = ut + \frac{1}{2}at^2$$
:

$$x = u \cos \theta t$$

$$y = u\sin\theta t - \frac{1}{2}gt^2$$

Now use the value we found earlier for total flight time, to give:

$$x = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}$$
 and here we can use $2\sin\theta\cos\theta \equiv 2\sin2\theta$

Hence,
$$x = \frac{u^2 \sin 2\theta}{g}$$
.

This result is known as the range, and it represents the horizontal distance covered. This value is a maximum when $\theta = 45^{\circ}$ and $\sin 2\theta = 1$. The angle at which the object is projected is known as the **angle of elevation**, and is measured upwards from the horizontal. Please note that the above results for the range and total flight time have been derived from the displacement and velocity components of a projectile, which are not in the formula booklet. It is therefore suggested that you know how to quickly derive these results when needed.

) TIF

Differentiating these two results gives $v_x = u\cos\theta$, $v_y = u\sin\theta - gt$. These are the same values we have already established.

WORKED EXAMPLE 13.1

A particle is projected from a point on a horizontal surface with initial speed $20\,\mathrm{m\,s^{-1}}$ at an angle of elevation of 25°. Find:

a the range of the particle

- b the time taken to reach the highest point
- c the speed of the particle when t = 0.5.

Answer

a Using $x = \frac{u^2 \sin 2\theta}{g}$, $x = \frac{20^2 \sin 50^\circ}{10}$

Substitute the values into the formula for the horizontal range.

Hence, $x = 30.6 \,\text{m}$.

 $t = 0.845 \,\mathrm{s}$

b At the highest point $v_y = 0$, so $t = \frac{u \sin \theta}{g} = \frac{20 \sin 25}{10}$ Using $v_y = u \sin \theta - gt$ leads to the time taken.

310

$$v_x = 20\cos 25 = 18.126$$

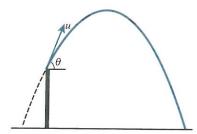
Determine both components of the speed at the given time.

$$v_y = 20\sin 25 - 10 \times 0.5 = 3.452$$

Find the magnitude of the resultant speed.

So the speed is $\sqrt{18.126^2 + 3.452^2} = 18.5 \,\mathrm{m \, s^{-1}}$.

Particles that are projected from raised platforms can be viewed as following part of a parabolic path.



This diagram shows a particle's trajectory from a raised point. Trace the path backwards to see where it would have started if it had been launched from ground level.

Consider an example where u=15, $\theta=60^\circ$, and the platform is 10 m above the ground. How would you go about finding the final speed of the projectile when it reaches ground level? Start with $v^2=u^2+2$ as vertically, so $v_y^2=(15\sin 60)^2-2\times 10\times (-10)$. This gives us $v_y=\frac{5\sqrt{59}}{2}$, which is the vertical component of the speed when the particle hits the ground.

Next use $v_x = 15\cos 60 = 7.5$, leading to $v = \sqrt{v_x^2 + v_y^2}$, which is equal to $5\sqrt{17} \,\mathrm{m\,s^{-1}}$.



TIP

When a particle is projected upwards, any position below the starting point is a *negative* displacement. Pay close attention to negative values in your calculations.

The question usually

states if the angle is

above the horizontal

(angle of elevation) or measured from the

vertical. When the

necessary.

angle is 45°, this is not

WORKED EXAMPLE 13.2

A particle is projected from the top of a platform 15 m above the horizontal floor below. The angle of projection is 45° and the initial speed is $25 \,\mathrm{m \, s^{-1}}$. Find:

- a the exact time taken to reach the horizontal floor below
- b the exact speed of the particle as it hits the floor.

Answer

a
$$s = ut + \frac{1}{2}at^2$$

Vertically, this gives:

$$-15 = 25t \sin 45 - 5t^2.$$

Rearranging we get:

$$5t^2 - \frac{25\sqrt{2}}{2}t - 15 = 0 \text{ or}$$
$$t^2 - \frac{5\sqrt{2}}{2}t - 3 = 0.$$

This is a quadratic equation, so using the quadratic formula we get:

$$t = \frac{2.5\sqrt{2} \pm 3.5\sqrt{2}}{2}.$$
So $t = 3\sqrt{2}$ s.

Model the vertical motion. Remember that the floor is 15 m below the starting position.

Calculate the two times; one is invalid as it is before the start of motion.

Note: The negative value for time represents the time when the particle would have started if it had been projected from ground level.

This answer is given in exact form as required by the question.

and vertically gives:

b Using v = u + at horizontally Find both components of the speed at the required time.

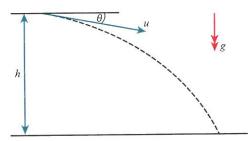
$$v_x = 25\cos 45 = 12.5\sqrt{2}$$

$$v_v = 25\sin 45 - 10 \times 3\sqrt{2}$$

=
$$-17.5\sqrt{2}$$

Hence, $v = \sqrt{v_x^2 + v_y^2} = 5\sqrt{37} \,\mathrm{m \, s^{-1}}$. Calculate the resultant in exact form.

Projectiles can, instead, be launched at an angle below the horizontal (angle of depression).



Let us consider a particle thrown with initial speed 40 m s⁻¹, at an angle of depression of 10°. Initially, it is at a height of 20 m above horizontal ground. To determine how far the particle travels in the horizontal direction, we must first find the time it takes to reach ground level, then apply this value of t to the horizontal displacement.

Using $s = ut + \frac{1}{2}at^2$ vertically, we have $20 = 40t\sin 10 + 5t^2$. Notice here we have taken the downwards direction as positive, so s = 20 and a = 10. This reduces the use of negatives so can make the work easier when working with an angle of depression.

This gives
$$5t^2 + 40t \sin 10 - 20 = 0$$
 so $t = \frac{-40\sin 10 \pm \sqrt{(-40\sin 10)^2 - 4(5)(-20)}}{2(5)}$

Solving this equation gives two values for t. The value we want is positive (the projectile starts at t = 0) so t = 1.423 s. Use $x = ut \cos \theta = 40 \cos 10 \times 1.423$, which gives a distance of 56.0 m.

WORKED EXAMPLE 13.3

An aircraft is on a scientific data collection mission over an ocean. The aircraft is travelling horizontally at a speed of 50 m s⁻¹ when it launches a sensor at an angle of 25° below the horizontal with an initial speed of 10 m s⁻¹. Given that the plane is 1 km above the ocean, and assuming that the ocean surface is flat, find the speed of the sensor when it is 100 m from the surface of the ocean.

Assume that air resistance is negligible.

Answer

Using $s = ut + \frac{1}{2}at^2$ vertically and taking downwards as positive:

$$900 = 10t \sin 25 + 5t^2.$$

Determine the time taken to fall 900 m.

Then
$$t^2 + 2t\sin 25 - 180 = 0 \Rightarrow t = 13$$
 Find the time taken.

So $v_y = 10\sin 25 + 10 \times 13 = 134.23$.

Calculate the vertical component of speed using

v = u + at vertically.

The plane is moving at the time of launch:

So $v_x = 50 + 10\cos 25 = 59.06$.

Find the horizontal component of speed using the initial speed of the plane and v = u + at horizontally.

Hence, $v = \sqrt{v_x^2 + v_y^2} = 147 \,\mathrm{m \, s^{-1}}.$

Find the resultant speed.

(A) KEY POINT 13.1

When an object is projected with an angle of depression, choose downwards as the positive direction. This means your vertical displacement is positive and the value of a is also positive.

EXERCISE 13A

- 1 A projectile is launched from point O on horizontal ground, landing at the point A.
 - a $OA = 205 \,\mathrm{m}$, $\theta = 40^{\circ}$. Find the speed u.
 - **b** $\theta = 30^{\circ}$, $u = 32 \,\mathrm{m \, s^{-1}}$. Find the range *OA*.
 - c $OA = 130 \,\mathrm{m}$, $u = 40 \,\mathrm{m \, s^{-1}}$. Find the possible angles of projection θ .
- A projectile is launched from a point on horizontal ground with speed *U*. Given that the horizontal distance travelled before hitting the ground again is 140 m, and that the angle of projection is 25° above the horizontal, find *U*.
 - 3 In a game a ball is kicked from the point O on horizontal ground, so that it lands on a scoring area which extends from 20 m to 25 m from O. If the angle of elevation when kicked is 35°, find the initial speed required for the ball to land in the scoring area.
- A particle is projected from point A on horizontal ground with initial speed $25 \,\mathrm{m\,s^{-1}}$ and angle of elevation θ. Given that the range of the particle is $60 \,\mathrm{m}$, find the value of the angle θ.
- A football player kicks a football from a point on the ground with an angle of elevation of 15°. The ball must land on horizontal ground level between 10 m and 20 m from the player's feet. Find the possible range of values of the initial speed of the football.
- A basketball is thrown with speed $10 \,\mathrm{m\,s^{-1}}$ from a point 2m above the ground, at an angle of 45° above the horizontal. Find the speed of the basketball when it is at a height of 4m for the second time during its motion.
- A particle of mass 2 kg is fired up a smooth slope of length 4 m, with initial speed 10 m s⁻¹, which is inclined at 30° above the horizontal.

The bottom of the slope is at the same level as horizontal ground.

- a Find the speed of the particle at the top of the slope.
- b The particle now flies off the slope and travels as a projectile. Find the greatest height achieved above the ground level.

- M A particle is projected horizontally from a point that is 12 m above a horizontal surface, with a speed of 15 m s⁻¹. Find the horizontal distance travelled before the particle hits the surface below.
- 9 A particle is projected across a horizontal area of land, with initial speed 30 m s⁻¹ and inclined at an angle of 40°. Find the duration of time for which the particle is at least 10 m above the ground.
- A small stone is thrown from the top of a building that is 30 m tall. The stone is given an initial speed of 5 m s⁻¹, and it is directed downwards with an angle of depression of 15°.
 - a Find the time taken for the stone to reach the ground.
 - b Find the speed of the stone as it hits the ground.

13.2 The Cartesian equation of the trajectory

In Section 13.1 we saw that the general motion of a projectile launched from ground level is governed by its initial speed and the angle at which it is projected.

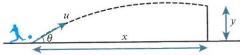
Consider the horizontal part of the motion. We know that $x = ut \cos \theta$. This means that, at any time, the value of t is given by $t = \frac{x}{u \cos \theta}$.

Now consider the vertical part of the motion. We know that $y = ut\sin\theta - \frac{1}{2}gt^2$. Substituting the expression found for t into this result gives $y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$.

Simplifying this, we get $y = x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta$.

This is essentially a quadratic equation in the form of an inverted parabola. The value of θ , once chosen, is constant and so only x and y vary. The equation is known as the Cartesian equation of the trajectory of a projectile.

Consider a ball that is to be kicked over a wall. The wall is 5 m tall and is 20 m from the starting position of the ball.



If the ball is kicked at an angle of 30° above the horizontal, how can we find the minimum speed required so that the ball just passes over the wall?

Begin with $y = x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta$. At the top of the wall, x = 20, y = 5.

So $5 = 20 \tan 30 - \frac{g \times 20^2}{2u^2} \sec^2 30$. This can be written as $5 = \frac{20\sqrt{3}}{3} - \frac{8000}{3u^2}$, from which we find $u = 20.2 \text{ m s}^{-1}$.

When a particle is at ground level during its motion, y = 0. This means $x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta = 0$ or $x \left(\tan \theta - \frac{g}{2u^2} x \sec^2 \theta \right) = 0$. This gives x = 0 and $x = \frac{2u^2 \tan \theta}{g \sec^2 \theta} = \frac{2u^2 \sin \theta \cos \theta}{g} = \frac{u^2 \sin 2\theta}{g}$,

the start and finish points of the motion, as shown in Key point 13.2.

O) KEY POINT 13.2

When
$$y = 0$$
, $x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta = 0$ or $x \left(\tan \theta - \frac{g}{2u^2} x \sec^2 \theta \right) = 0$.

So the start and finish points of the motion are x = 0 and $x = \frac{u^2 \sin 2\theta}{\sigma}$.

WORKED EXAMPLE 13.4

A projectile is launched from a point O, 1 m above horizontal ground level. It is given an initial speed of $20 \,\mathrm{m\,s^{-1}}$ and an angle of elevation of 25°. Find the horizontal distance from O when the particle is 2 m above the ground and descending.

Answer

Start with y = 1. Use $y = x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta$. The projectile starts at 1 m above the ground, so y = 1 for the particle to be 2 m above the ground.

Then $1 = x \tan 25 - \frac{10x^2}{2 \times 20^2} \sec^2 25$ or $\frac{\sec^2 25}{80} x^2 - x \tan 25 + 1 = 0$ leads to two results.

Input the known values into the trajectory equation.

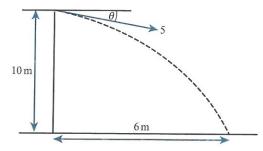
These are 2.32 and 28.3.

Obtain two results for x. One value is when the projectile is ascending and the other value is when it is descending.

Hence, $x = 28.3 \,\text{m}$.

Choose the larger value of x for when the projectile is descending.

Consider a particle being projected downwards instead. Remember that gravity assists when particles are descending, so it is a good idea to make the downward acceleration positive.



For example, consider a particle projected from a point 10 m above horizontal ground, with initial speed $5\,\mathrm{m\,s^{-1}}$ and an angle of depression θ . Can we determine the angle if the horizontal distance travelled before hitting the ground is 6 m?

Start with the equation $y = x \tan \theta + \frac{gx^2}{2u^2} \sec^2 \theta$, giving $10 = 6 \tan \theta + \frac{10 \times 6^2}{2 \times 5^2} (1 + \tan^2 \theta)$,

which leads to $7.2 \tan^2 \theta + 6 \tan \theta - 2.8 = 0$. From here, $\tan \theta = \frac{1}{3}$ or $\tan \theta = -\frac{7}{6}$. However, $\tan \theta = -\frac{7}{6}$ would give a negative value for θ . The magnitude of this angle would represent

WORKED EXAMPLE 13.5

A ball is thrown from the top of a building of height 20 m. The ball is thrown such that the initial speed is $U \text{m s}^{-1}$ and the angle of depression of the throw is 30°. Find:

- a the horizontal distance travelled in terms of U
- **b** the horizontal distance when U = 5.

Answer

a Use $y = x \tan \theta + \frac{gx^2}{2u^2} \sec^2 \theta$. Use the Cartesian equation of the trajectory.

So $20 = \frac{x\sqrt{3}}{3} + \frac{20x^2}{3U^2}$. Substitute the values for the angle and height.

So $20x^2 + \sqrt{3}U^2x - 60U^2 = 0$. Rearrange to get a quadratic in x.

Then $x = \frac{1}{40} \sqrt{3} \left(-U^2 + \sqrt{U^2(1600 + U^2)} \right)$ m. Solve for x. (Ignore the negative solution.)

b When U = 5, x = 7.65 m. Find the solution when U = 5.

Consider a particle that follows a parabolic path, such as $y = 0.5x - 0.01x^2$. When we compare this equation with the trajectory, $y = x \tan \theta - \frac{gx^2}{2u^2} \sec^2 \theta$, it is clear that $\tan \theta = 0.5$ and also that $\frac{10}{2u^2} \sec^2 \theta = 0.01$.

To solve these we must first find $\theta = 26.57^{\circ}$ then, using $\sec^2 \theta = 1 + \tan^2 \theta$, we obtain $\sec \theta = \frac{\sqrt{5}}{2}$ and $u = 25 \,\mathrm{m \, s^{-1}}$.

WORKED EXAMPLE 13.6

A projectile follows the path $y = 0.3x - 0.1x^2$. Find the initial speed and the angle of elevation of the particle.

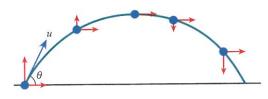
Answer

 $\tan \theta = 0.3$, so $\theta = 16.70^{\circ}$. Compare the coefficients for the linear term.

 $\frac{g}{2u^2}\sec^2\theta = 0.1$ Compare the coefficients for the quadratic term and use $\sec^2\theta = 1 + \tan^2\theta$ to find the value of $\sec\theta$.

Since $\sec^2 \theta = 1 + 0.3^2$, it follows that $u^2 = \frac{10 \times 1.09}{2} \times 10$. So $u = 7.38 \,\mathrm{m \, s^{-1}}$.

We will now look at the direction of motion of a projectile at certain points in its path.



As the object follows its path the horizontal component of velocity is unchanged. However, the vertical component goes from its greatest positive value initially to zero at the highest point of the trajectory to its greatest negative value when it reaches the ground again.

This implies that the direction of motion of the particle is related to its velocity at that time. To find the direction we just need to know the components of velocity. We already know the horizontal component since it is constant throughout the motion. So we only need to find the vertical component.

Consider a particle with initial speed $20 \,\mathrm{m\,s^{-1}}$, projected at an angle of elevation of 50° from horizontal ground. Is it possible to find the height of the particle when it is travelling in a direction of 45° below the horizontal?

When the angle is 45° below the horizontal, from the diagram we see that $v_x = -v_y$, and so $20\cos 50 = -(20\sin 50 - gt)$. This means we can work out the time at this point and use this time to find the height. So $t = \frac{20}{g}(\cos 50 + \sin 50) = 2.818$ s. Then, using $y = 20t\sin 50 - 5t^2$ gives a height of 3.47 m.



WORKED EXAMPLE 13.7

A particle is projected from the top of a tower that is 50 m tall. The initial speed is 25 m s⁻¹ and the angle of depression is 10°. Find the height of the particle above the ground when the downward angle of the direction of the particle is 30°.

Answer

The tangent of the angle is $\frac{v_y}{v_x}$.

Note that $\tan 30 = \frac{v_y}{v_x}$. Then we have $v_y = \frac{\sqrt{3}}{3}v_x$.

Recognise the relationship between the tangent and the components.

So $25\sin 10 + gt = \frac{\sqrt{3}}{3}(25\cos 10)$.

Use this relationship with $v_y = 25 \sin 10 + gt$ and $v_x = 25 \cos 10$. Note that this time v_y is positive since the motion is downwards throughout.

From this $t = 0.9873 \, \text{s}$.

Solve to find the time.

Then $y = 25t \sin 10 + 5t^2$ gives y = 9.1603.

Work out the vertical distance travelled from the point of release.

Hence, $50 - y = 40.8 \,\mathrm{m}$ above the ground.

Determine the height.

$oldsymbol{1}$ did you know?

Galileo rolled inked bronze balls down an inclined plane to determine where a projectile would land. This experiment allowed Galileo to determine that the path of a projectile is very close to being parabolic.

EXERCISE 13B

- 1 A particle is projected from horizontal ground with initial speed 15 m s⁻¹ and angle of elevation 60°. Find the horizontal distance travelled when the particle is first at a height of 8 m above ground level.
- 2 A ball is thrown from a point 2m above horizontal ground. The initial speed is $20 \,\mathrm{m\,s^{-1}}$ and the angle of elevation is 45° . Find the horizontal distance covered when the ball is $12\,\mathrm{m}$ above the ground.
- 3 A particle is projected from ground level with initial speed 30 m s⁻¹ at an angle of elevation of 25°. Find the horizontal distance travelled from the starting point when the height is 4 m for the first time.
- A particle is projected from a point 5 m above a horizontal plane. The angle of elevation is 10°. Given that the particle travels 75 m before hitting the ground, find the initial speed.
- 5 A particle is projected from a platform 4m above horizontal ground, with an initial speed of 20 m s⁻¹ and an angle of elevation of 20°. Find the direction of the particle as it lands on the ground.
- M PS 6 A ball is kicked at a house window. The window is 4 m up a vertical wall from a horizontal floor. The ball is kicked from a position that is 12 m distance from the foot of the wall. If the ball enters the window at an angle that is 30° below the horizontal, find:
 - a the time taken to reach the window
 - b the initial speed of the ball.
 - 7 A particle is projected from the top of an office building that is 35 m tall. The initial speed is 14 m s⁻¹ and the angle of depression is 20°. Find the height of the particle above the ground when the downward angle of the direction of the particle is 45°.
 - P A particle is projected from a point on a horizontal surface with initial speed u and angle of elevation β . At any time during its motion, the particle is at the position (x, y), where x is the horizontal distance travelled and y is the vertical distance travelled.

Show that
$$y = x \tan \beta - \frac{gx^2}{2u^2} (1 + \tan^2 \beta)$$
.

- 9 A particle is projected from a point on a horizontal plane with speed $20 \,\mathrm{m\,s^{-1}}$ at an angle of elevation of θ . Given that the particle passes through the point x = 20, y = 10, find the possible angles of elevation.
- P 10 A particle is projected from a point on horizontal ground with initial speed u and angle of elevation α . Show that $x^2 + y^2 = u^2t^2 10yt^2 25t^4$, where (x, y) is the particle's position at time t.



Consider $\cos^2 \theta + \sin^2 \theta$.

TIP

WORKED EXAM-STYLE QUESTION

A small stone is projected from a point O on horizontal ground, with speed $V \text{m s}^{-1}$ at an angle θ° above the horizontal. The horizontal and upward vertical displacements of the particle from O at time t seconds after projection are x and y respectively. The equation of the stone's trajectory is $y = 0.75x - 0.02x^2$, where x and y are in metres. Find:

- a the values of θ and V
- **b** the distance from O of the point where the stone hits the ground
- c the greatest height reached by the stone.

Answer

a Comparing $y = x \tan \theta - \frac{gx^2}{2V^2} \sec^2 \theta$ with $y = 0.75x - 0.02x^2$, it is clear that $\tan \theta = \frac{3}{4}$. Hence, $\theta = 36.9^\circ$.

Then $\frac{g}{2V^2} \sec^2 \theta = 0.02$, and since $\cos \theta = \frac{4}{5}$, $\sec^2 \theta = \frac{25}{16}$. Hence, $V = \frac{25\sqrt{10}}{4} = 19.8 \,\text{ms}^{-1}$.

- **b** Letting y = 0, $0.75x 0.02x^2 = 0$. So x = 0 or 0.02x = 0.75. Hence, x = 37.5 m.
- c At the highest point $x = \frac{37.5}{2} = 18.75$, so $y = 0.75 \times 18.75 0.02 \times 18.75^2 = 7.03$ m.

Checklist of learning and understanding

Governing equations:

- For displacement: $x = ut \cos \theta$ and $y = ut \sin \theta \frac{1}{2}gt^2$
- For velocity: $v_x = u\cos\theta$ and $v_y = u\sin\theta gt$
- For the range: $x_{\text{max}} = \frac{u^2 \sin 2\theta}{g}$
- Total flight time: $t = \frac{2u\sin\theta}{g}$

Cartesian equation of the trajectory:

- $y = x \tan \theta \frac{gx^2}{2u^2} \sec^2 \theta$
- If given as $y = ax bx^2$, then $a = \tan \theta$ and $b = \frac{g}{2u^2} \sec^2 \theta$.

Direction of motion:

- Particles projected upwards are subject to -g, so $y = ut \sin \theta \frac{1}{2}gt^2$ and $v_y = u \sin \theta gt$.
- Particles projected downwards have +g, and so $y = ut\sin\theta + \frac{1}{2}gt^2$ and $v_y = u\sin\theta + gt$.
- For the angle of projection use $\tan \theta = \frac{v_y}{v_x}$.





The formulae for range and total flight time apply only to particles that start and finish their motion on the same horizontal level.

END-OF-CHAPTER REVIEW EXERCISE 13

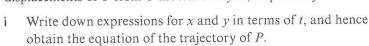


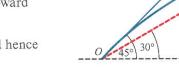
- A particle is projected from a point O on horizontal ground. The velocity of projection has magnitude $20 \,\mathrm{m\,s^{-1}}$ and direction upwards at an angle θ to the horizontal. The particle passes through the point, which is $7 \,\mathrm{m}$ above the ground and $16 \,\mathrm{m}$ horizontally from O, and hits the ground at the point A.
 - i Using the equation of the particle's trajectory and the identity $\sec^2 \theta = 1 + \tan^2 \theta$, show that the possible values of $\tan \theta$ are $\frac{3}{4}$ and $\frac{17}{4}$.
 - ii Find the distance OA for each of the two possible values of $\tan \theta$.
 - iii Sketch in the same diagram the two possible trajectories.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q5 June 2010



A particle P is projected from a point O with initial speed $10 \,\mathrm{m\,s^{-1}}$ at an angle of 45° above the horizontal. P subsequently passes through the point A, which is at an angle of elevation of 30° from O (see diagram). At time ts after projection the horizontal and vertically upward displacements of P from O are x m and y m, respectively.





- ii Calculate the value of x when P is at A.
- iii Find the angle the trajectory makes with the horizontal when P is at A.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q7 November 2010



- A particle P is projected with speed $35 \,\mathrm{m\,s^{-1}}$ from a point O on a horizontal plane. In the subsequent motion, the horizontal and vertically upwards displacements of P from O are x m and y m, respectively. The equation of the trajectory is $y = kx \frac{(1+k^2)x^2}{245}$, where k is a constant. P passes through the points A(14,a) and B(42,2a), where a is a constant.
 - i Calculate the two possible values of k and, hence, show that the larger of the two possible angles of projection is 63.435°, correct to 3 decimal places.

For the larger angle of projection, calculate

- ii The time after projection when P passes through A,
- iii The speed and direction of motion of P when it passes through B.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q7 November 2016



Chapter 14

Equilibrium of a rigid body

In this chapter you will learn how to:

- find the moment of a force and use it to determine the centre of mass of 2- and 3-dimensional shapes
- determine the centre of mass of a composite body, made up of standard shapes
- make use of the fact that the vector sum of forces is zero when objects are in equilibrium
- solve problems with coplanar forces, including objects on the point of sliding or toppling.

321

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Mechanics, Chapter 2	Resolve forces in two perpendicular directions.	1 An object of mass m is sliding down a smooth slope inclined at an angle θ to the horizontal. Find the components of the weight of the object parallel and perpendicular to the slope.
AS & A Level Mathematics Mechanics, Chapter 4	Be able to determine or make use of the coefficient of friction of an object on the point of moving.	2 A particle of mass 2 kg is resting on a rough slope inclined at an angle θ to the horizontal. The maximum force that the frictional force can produce is 10 N . Find the value of θ for which the particle is on the point of slipping.

What is equilibrium?

When all forces acting on an object are balanced and the vector sum of the forces is zero, we say the object is in a state of **equilibrium**. Equilibrium is one of the most important concepts in engineering analysis. It allows you to check whether systems are stable and to calculate otherwise unknown forces.

In this chapter, we shall find the moment, or turning effect of a force, produced when we apply a force at a perpendicular distance to an object. For moments to be applied, the object must have length, so we cannot model it as a particle. Instead, we use a rigid body, a larger version of a particle. This is assumed to be inflexible so it does not bend when forces are applied.

We shall use this to find the centre of mass of standard shapes and **composite** bodies. Finally, these bodies will be placed in positions such that they are on the point of **breaking equilibrium** by sliding or toppling. To tackle the problems, we will resolve forces and take moments. The symbols we use are shown in Key point 14.1.

(A) KEY POINT 14.1

In this chapter, you will use the symbols a for acceleration, v for velocity, and x for displacement. Unless stated otherwise, $g = 10 \,\mathrm{m\,s^{-2}}$.

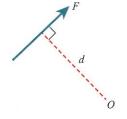
14.1 The moment of a force

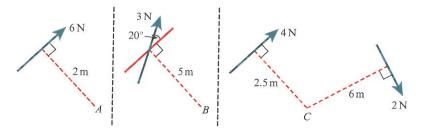
Imagine trying to loosen a nut with a spanner (wrench). Applying the force with a short spanner would make it difficult to loosen the nut. But applying the same force with a longer spanner will result in a greater turning effect. This turning effect is known as a moment.

Moments are calculated by multiplying a force by a perpendicular distance, as shown in Key point 14.2. This means that moments are to be measured in newton metres (Nm). When we talk about moments, we refer to the moment of that particular force.

(D) KEY POINT 14.2

The moment of a force, F, about a point O is $F \times d$, where d is the perpendicular distance from the point O to the line of action of the force F. If the distance between the force and the point O is zero, there is no turning effect.





From the diagram, the moment of the 6N force about the point A can be calculated as $6 \times 2 = 12 \text{ Nm}$.

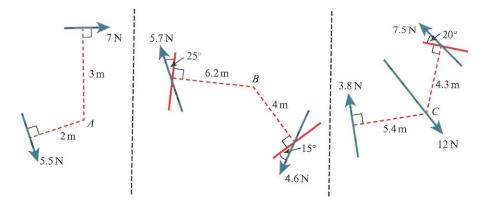
For the moment of the force about B, we first need to find the component of the force that is perpendicular to the line through B. So the moment is $3\cos 20 \times 5 = 14.1 \,\mathrm{Nm}$.

The other component of the force, $3\sin 20$, passes through B, so its moment is zero.

For the moment of the forces about the point C, we add the turning effects since they both act in the same direction, which is clockwise. So the moment is $4 \times 2.5 + 2 \times 6 = 22 \text{ N m}$.

WORKED EXAMPLE 14.1

For each of the following cases, work out the moment and state whether it is a clockwise or anticlockwise turning effect.



Answer

For $A \circlearrowleft : 7 \times 3 - 5.5 \times 2 = 10 \text{ Nm clockwise}$

Choose clockwise or anticlockwise as your positive direction.

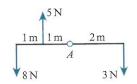
For B \circlearrowleft : 5.7 cos 25 × 6.2 + 4.6 cos 15 × 4 = 49.8 N m clockwise

Determine the components of forces.

For C \bigcirc : $3.8 \times 5.4 - 7.5 \cos 20 \times 4.3 = -9.79$, therefore 9.79 N m anticlockwise.

It is better to state a positive moment 'anticlockwise', rather than a negative moment clockwise.

If we take a rod and apply a series of forces to it, we can observe the turning effect on the rod. We generally model it as a 'light' rod, so we can ignore the mass, and a **rigid body**, so it does not bend when forces are applied.

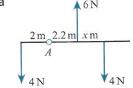


So taking moments about the point A, $O: 3 \times 2 + 5 \times 1 - 8 \times 2 = 5 \text{ Nm}$ anticlockwise. The 5 N force and the 3 N force turn the same way about the point A, whereas the 8 N is in the opposite direction. This means that the rod will turn about A as it is not in equilibrium.

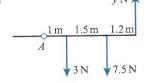
WORKED EXAMPLE 14.2

In each case, a light rod is pivoted at a fixed point, A. Find the unknown values such that the rod will have a zero moment about A.

a



b



2.1 m 0.5 m 1.7 m

Answer

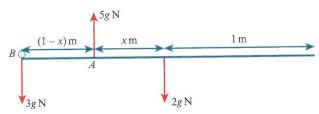
a About $A \circlearrowleft : 4 \times (x + 2.2) - 6 \times 2.2 - 4 \times 2 = 0$ So 4x + 8.8 = 21.2, giving x = 3.1 m. Note the distance of x + 2.2 between the clockwise force and A.

b About $A \circlearrowleft: 3 \times 1 + 7.5 \times 2.5 - y \times 3.7 = 0$ So 3.7y = 21.75, giving y = 5.88 N. Make sure the total moment is zero in order to determine *y*.

c About $A \odot: 4.8 \times 4.3 + 2.5 \times 1.7 - 16\cos\theta \times 2.2 = 0$ So $35.2\cos\theta = 24.89$, giving $\theta = 45.0^{\circ}$.

Remember to use only the component of the force perpendicular to the rod.

Consider a uniform rod of length $2 \,\mathrm{m}$ and mass $2 \,\mathrm{kg}$ resting in equilibrium over the edge of a table. The edge of the table is the point A. The mass B at the end of the rod is $3 \,\mathrm{kg}$.



This time we consider the weight of the rod. Because the rod is modelled as uniform, its weight acts at the centre of the rod.

If we ignore the small mass at the point B, taking moments about A gives $2g \times x = 20x$ N m. This means the rod would not be in equilibrium and would turn about the point A.

So we include the mass 3 g and take moments again, now 2gx = 3g(1 - x), and so $x = \frac{3}{5}$ m. The reaction force has a moment of zero about the point A.

WORKED EXAMPLE 14.3

A uniform rod, AB, of length 5 m and mass 10 kg, is placed over the edge of a cliff such that B is 4 m from the edge of the cliff and hanging over the edge. A man, of mass 80 kg, stands on the cliff side of the rod, a distance of x m from the edge of the cliff.

a Find the value of x so that the rod is on the point of tipping over the cliff.

The man now stands at A and a boy of mass 35 kg walks across the rod towards B.

b Can the boy walk all the way to the end of the rod?

Answer

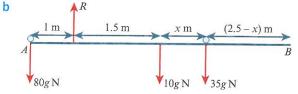
a R 1.5 m 2.5 m

About edge \circlearrowleft : $10g \times 1.5 = 80g \times x$, so $x = \frac{3}{16}$ m.

Sketch the first case, labelling your diagram fully.

Taking moments about the edge gives x.

Redraw diagram with the boy included and the man now at point A.



About edge \circlearrowleft : $10g \times 1.5 + 35g \times (1.5 + x) = 80g \times 1$

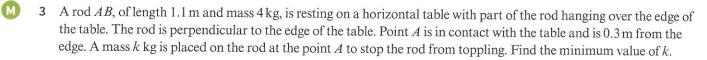
Then $1.5 + x = \frac{80g - 15g}{35g}$. So, at most, $x = \frac{5}{14}$ m.

Taking moments about the edge shows that the distance is less than 2.5 m, so the boy can never reach *B*.

So the boy cannot reach the end of the rod at B.

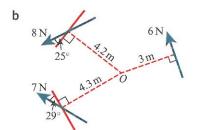
EXERCISE 14A

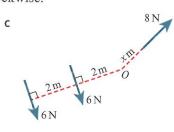
- 1 A rod AB of length 3.1 m and mass 8 kg is balancing on a pivot at a point C, where AC is 1.2 m. It is kept balanced by masses being placed at A and B. If the mass at A is 4 kg, determine the mass at B.
- 2 A rod AB of length 1.4m and mass 6 kg is balancing on a pivot at a point C, where AC is 0.5 m. It is kept balanced by masses being placed at A and B. If the mass at A is 3 kg, determine the mass at B.



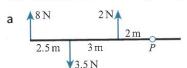
4 Find the moment about O of the forces shown, stating if it is clockwise or anticlockwise.

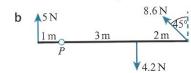
a 3N 2m 2N 5N 2N

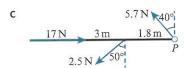




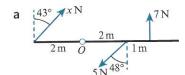
325

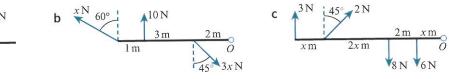


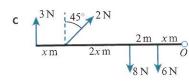




- 6 A rod, AB, of length 0.4 m and mass 5 kg, is resting such that part of the rod is hanging over the edge of a horizontal table. The rod is positioned so that it is perpendicular to the edge of the table, and with the point A in contact with the table and 0.15m from the edge.
 - A mass of 8 kg is placed on the rod to stop the rod from falling off the table. Find its distance from the point A.
 - **b** The 8 kg mass is now placed at the point A. A second mass is added at the point B. Find the maximum mass that can be added without the rod toppling.
- 7 A rod, AB, of length 5 m and mass 12 kg is hanging over the edge of a cliff. The end A is 1.2 m from the edge of the cliff, and the rod is assumed to be perpendicular to the cliff. A woman, of mass M, stands at the point A, and a girl, of mass m, stands at the point B. Given that the rod is on the point of tipping, find m in terms of M.
- In each case, find the unknown such that the total moment about O is zero.





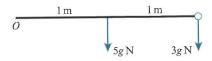


326

14.2 Centres of mass of rods and laminas

We can use moments in finding the centre of mass of an object. All the weight of an object acts through its centre of mass.

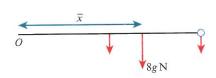
For example, consider a rod of length 2m and mass 5kg. We add a mass of 3kg to one end.



If we take moments about $O \circlearrowleft : 5g \times 1 + 3g \times 2 = 11g \text{ N m}$.

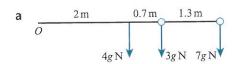
Now consider this turning effect as coming from one force positioned where $8g \times \overline{x} = 11g$, then $\overline{x} = \frac{11}{9}$ m from the point O.

We have created a single force a distance of $\frac{11}{8}$ m from O. This force multiplied by the distance \bar{x} represents the sum of all the turning effects of the system.

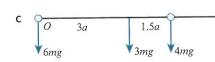


WORKED EXAMPLE 14.4

Find the distance, \bar{x} , of the single force that represents all other forces from the point O.







Answer

a About $O \circlearrowleft : 4g \times 2 + 3g \times 2.7 + 7g \times 4$ $= 44.1g = 14g \times \bar{x}$

Take moments about O, then compare this to the sum of the forces $\times \bar{x}$.

Hence, $\bar{x} = 3.15 \,\mathrm{m}$.

c About $O \circlearrowleft : 3mg \times 3a + 4mg \times 4.5a$ = $27mga = 13mg \times \overline{x}$

Although the moment of the 6mg force at O is zero, you must remember to include the weight when finding \bar{x} .

Hence, $\overline{x} = \frac{27}{13}a$.

$oldsymbol{\wp})$ KEY POINT 14.3

The distance of the centre of mass \bar{x} from the point of reference is equal to $\frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$, where m_i is the mass of each element of the object, and x_i is the distance of each element from a point of reference about which moments are taken.

Let us look at a 2-dimensional case such as a framework with masses added.

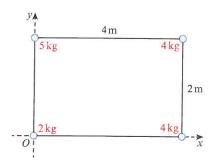
We can now work out the centre of mass in two dimensions. This will give us coordinates relative to two perpendicular axes of reference. In this framework, we shall assume each rod is light but there are masses attached at each corner.

To take moments about a point, we assume the framework is horizontal and the weights have a turning effect on the framework.

Taking moments about $Ox \circlearrowleft: 5g \times 2 + 4g \times 2 = 15g \times \overline{y}$, so $\overline{y} = 1.2 \text{ m}$. Remember that the masses that lie along Ox will not contribute to the turning effect.

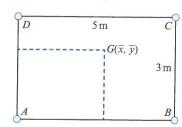
Then moments about Oy $\circlearrowleft: 4g \times 4 + 4g \times 4 = 15g \times \overline{x}$, so $\overline{x} = \frac{32}{15}$ m.

So the centre of mass G has position $\left(\frac{32}{15}, 1.2\right)$ relative to the point O.



WORKED EXAMPLE 14.5

A rectangular framework, ABCD, is made of four light rods. There are masses on the rods at the given points.



In each case, determine the coordinates of the centre of mass. Use AB and AD as your axes, with A as your origin.

- a 2kg at A, 4kg at B, 7kg at C, 2.5kg at D
- **b** 5 kg at the midpoint of AD, 6 kg at B, 4 kg at E, where $CE = \frac{1}{3}CB$
- c 6 kg at A, 3 kg at the midpoint of AB, 8 kg at B, 10 kg at point F, where $DF = \frac{3}{5}DC$

a About $AB \circlearrowleft : 7g \times 3 + 2.5g \times 3 = 15.5 \times \overline{y}$ Ignore the masses at A and B for this part.

So
$$\bar{y} = \frac{57}{31}$$
.

About $AD \circlearrowleft : 4g \times 5 + 7g \times 5 = 15.5 \times \overline{x}$ Ignore the masses at A and D.

So
$$\bar{x} = \frac{110}{31}$$
.

Hence, centre of mass is $G\left(\frac{110}{31}, \frac{57}{31}\right)$. State the centre of mass.

b About $AB \circlearrowleft : 5g \times 1.5 + 4g \times 2 = 15g \times \overline{y}$ Note that the mass at A is zero.

So
$$\bar{y} = \frac{31}{30}$$
.

About AD \circlearrowleft : $6g \times 5 + 4g \times 5 = 15g \times \overline{x}$

So
$$\bar{x} = \frac{10}{3}$$
.

Hence, centre of mass is $G\left(\frac{10}{3}, \frac{31}{30}\right)$.

c About $AB \circlearrowleft : 10g \times 3 = 27g \times \overline{y}$

So
$$\bar{y} = \frac{10}{9}$$
.

About AD O:

 $3g \times 2.5 + 8g \times 5 + 10g \times 3 = 27g \times \overline{x}$ Remember that DF = 3 m.

So
$$\bar{x} = \frac{155}{54}$$
.

Hence, centre of mass is $G\left(\frac{155}{54}, \frac{10}{9}\right)$.

We shall now consider a thin, 2-dimensional shape known as a lamina. If a lamina is described as uniform, then we assume that its mass is spread evenly across its area.

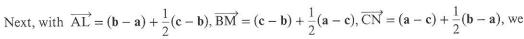
The standard shapes you will encounter are the rectangle, circle, triangle and sector of a circle.

For a rectangle the centre of mass is in the centre, where the lines of symmetry meet.

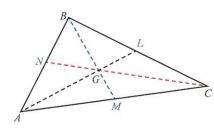
For triangles, consider the scalene triangle ABC, with L, M, N at the midpoints of BC, AC and AB, respectively.

Now, relative to an origin O, we have $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. We want to find \overrightarrow{OG} .

First, state that $\overrightarrow{OG} = \overrightarrow{OA} + \alpha \overrightarrow{AL}$, or $\overrightarrow{OG} = \overrightarrow{OB} + \beta \overrightarrow{BM}$, or $\overrightarrow{OG} = \overrightarrow{OC} + \gamma \overrightarrow{CN}$. The scalars α, β, γ are all between 0 and 1.



have
$$\overrightarrow{OG} = (1 - \alpha)\mathbf{a} + \frac{\alpha}{2}(\mathbf{b} + \mathbf{c}), \overrightarrow{OG} = (1 - \beta)\mathbf{b} + \frac{\beta}{2}(\mathbf{a} + \mathbf{c})$$
 and $\overrightarrow{OG} = (1 - \gamma)\mathbf{c} + \frac{\gamma}{2}(\mathbf{a} + \mathbf{b}).$

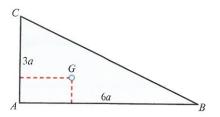


This means that each result must have the same coefficients for \mathbf{a} , \mathbf{b} , \mathbf{c} . For example, we see that $1 - \alpha = \frac{\beta}{2} = \frac{\gamma}{2}$ and $1 - \beta = \frac{\alpha}{2} = \frac{\gamma}{2}$. So $1 - \alpha = \frac{\alpha}{2}$, which leads to $\alpha = \beta = \gamma = \frac{2}{3}$

Hence,
$$\overrightarrow{OG} = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

Next, consider the triangle ABC with AB = 6a and AC = 3a. The centre of mass G can be found using A as the origin and then moving one-third of the distance along each edge to get G(2a, a).

Note that this is relative to the edges AB and AC. An alternative way of doing this is to use the result $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$, as shown in Key point 14.4, where the x_i and y_i terms are the vertices measured from an origin. In our case we used A as the origin, so $G\left(\frac{0+6a+0}{3}, \frac{0+0+3a}{3}\right)$ leads to the same result as before.



(D) KEY POINT 14.4

The centre of mass of any triangular lamina is given by $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, where the three vertices of the triangle are at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

WORKED EXAMPLE 14.6

For each case, work out the centre of mass of the triangle, stating your point of reference.

Answer

a Using AB and AC as axes, $\frac{1}{3} \times 10a$ and $\frac{1}{3} \times \frac{9}{2}a$ gives $G\left(\frac{10}{3}a, \frac{3}{2}a\right)$.

Set the axes. Then take one-third of each length from A.

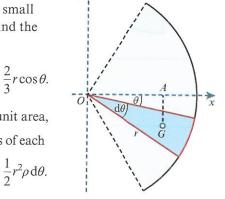
b Using O as the origin, $\left(\frac{1+3+8}{3}, \frac{1+5+2}{3}\right)$ gives $G\left(4, \frac{8}{3}\right)$.

Take the mean of x and y.

Consider a lamina in the shape of a sector of a circle with an angle 2α at the centre, a radius r and a centre O. Then a smaller sector is chosen. Its angle is $d\theta$, which is so small that the sector is almost a triangle, meaning we can use our results from above to find the position of the centre of mass of this small sector.

For that small sector, $OG = \frac{2}{3}r$, which means that the length $OA = \frac{2}{3}r\cos\left(\theta + \frac{1}{2}d\theta\right) \approx \frac{2}{3}r\cos\theta$.

The total mass of the large sector, represented by the area of the sector \times the mass per unit area, is $\frac{1}{2}r^2 \times 2\alpha \times \rho = r^2\alpha\rho$, where ρ , the Greek letter rho, is the mass per unit area. The mass of each small sector is $\frac{1}{2}r^2\rho \,\mathrm{d}\theta$, so summing each moment contribution, $r^2\alpha\rho\overline{x} = \int_{-\alpha}^{\alpha} \frac{2}{3}r\cos\theta \times \frac{1}{2}r^2\rho \,\mathrm{d}\theta$.



So $r^2 \alpha \rho \overline{x} = \frac{1}{3} \rho r^3 [\sin \theta]_{-\alpha}^{\alpha}$, leading to $\overline{x} = \frac{r(\sin \alpha - \sin(-\alpha))}{3\alpha} = \frac{2r \sin \alpha}{3\alpha}$, as shown in

Key point 14.5. This result can be used for any sector, giving the distance OG, where OG bisects the angle 2α subtended at the centre of the circle. The angle must be given in radians, not degrees.

(D) KEY POINT 14.5

The centre of mass of a sector-shaped lamina with an angle of 2α subtended at the centre is

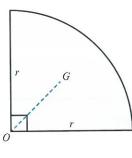
WORKED EXAMPLE 14.7

Determine the centre of mass from the centre O of a sector of radius r with angle:

a

 $b \pi$

Answer



$$OG = \frac{2r\sin\alpha}{3\alpha},$$

Quote the result.

here
$$\alpha = \frac{\pi}{4}$$
.

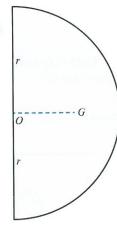
Take half the angle subtended at the centre.

Hence,
$$OG = \frac{2r\sin\frac{\pi}{4}}{\frac{3\pi}{4}}$$

Determine the distance OG.

which is $\frac{4r\sqrt{2}}{3\pi}$.

b



Again, using

$$OG = \frac{2r\sin\alpha}{3\alpha}$$
 with

 $\alpha = \frac{\pi}{2}$ gives the result

$$OG = \frac{2r\sin\frac{\pi}{2}}{\frac{3\pi}{2}},$$
which is $\frac{4r}{3\pi}$.

Since a semi-circle is a sector the result for the sector can be used with $a = \frac{\pi}{2}$.

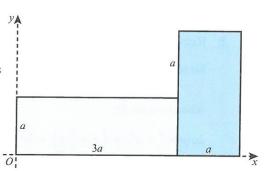
Don't forget that the angle must be in radians for this to work.

Now that we have seen some standard lamina, we can look at combining these shapes to form composite bodies. In this example we have a composite lamina made of two uniform rectangular laminae. Using ρ as the mass per unit area, one rectangle has mass $3a^2\rho$ and the other has mass $2a^2\rho$, so the whole body has mass $5a^2\rho$.

Taking moments about the y-axis $\circlearrowleft: 3a^2\rho \times \frac{3}{2}a + 2a^2\rho\left(3a + \frac{a}{2}\right) = 5a^2\rho\overline{x}$, and so $\bar{x} = 2.3a$.

Then taking moments about the x-axis $\circlearrowleft: 3a^2\rho \times \frac{a}{2} + 2a^2\rho \times a = 5a^2\rho \overline{y}$, and so $\bar{y} = 0.7a$.

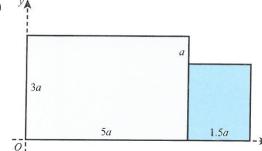
So the centre of mass is G(2.3a, 0.7a).

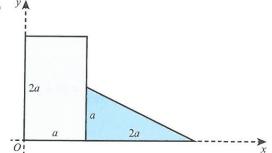


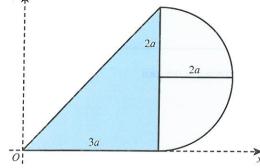
WORKED EXAMPLE 14.8

For each composite shape, determine the centre of mass relative to the axes shown.

a







Answer

a Large rectangle has mass $15a^2\rho$, small rectangle has mass $3a^2\rho$

So total mass is $18a^2\rho$.

About the y-axis O:

$$15a^{2}\rho \times \frac{5}{2}a + 3a^{2}\rho \left(5a + \frac{3}{4}a\right) = 18a^{2}\rho \bar{x}$$

So
$$\bar{x} = \frac{73}{24}a$$
.

About the x-axis
$$\circlearrowright$$
:

$$15a^2\rho \times \frac{3}{2}a + 3a^2\rho \times a = 18a^2\rho \overline{y}$$

So
$$\bar{y} = \frac{17}{12}a$$
.

Work out the mass of each part first. Use ρ as the mass per unit area. You can then find the total mass.

Take moments about each axis to determine the coordinates of the centre of mass. The height of the smaller rectangle must be 2a.

b Rectangle mass is $2a^2\rho$, triangle mass is $a^2\rho$ So total mass is $3a^2\rho$. For the triangular lamina the centre of mass is $\frac{1}{3}$ of the distance from the bottom left corner of the triangle.

About y-axis O:

$$2a^{2}\rho \times \frac{a}{2} + a^{2}\rho \times \left(a + \frac{2}{3}a\right) = 3a^{2}\rho \overline{x}$$

This gives $\overline{x} = \frac{8}{9}a$.

About x-axis \circlearrowright : $2a^2\rho \times a + a^2\rho \times \frac{a}{3} = 3a^2\rho \overline{y}$

This gives $\overline{y} = \frac{7}{9}a$.

c Triangle mass is $6a^2\rho$, semicircle mass is $2\pi a^2\rho$ So total mass is $(6 + 2\pi)a^2\rho$.

Centre of mass of semicircle from the vertical diameter shown is $\frac{4 \times 2a}{3\pi} = \frac{8a}{3\pi}$.

The height of the triangle is twice the radius of the semicircle.

For the semicircle we know that $OG = \frac{4r}{3\pi}$. We can quote this result and use it.

About y-axis O:

$$6a^{2}\rho \times 2a + 2\pi a^{2}\rho \times \left(3a + \frac{8a}{3\pi}\right) = (6 + 2\pi)a^{2}\rho \overline{x}$$

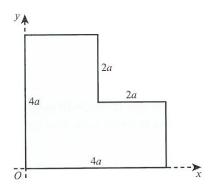
So $\bar{x} = 2.95a$.

About x-axis \circlearrowright :

$$6a^2\rho \times \frac{4}{3}a + 2\pi a^2\rho \times 2a = (6 + 2\pi)a^2\rho \bar{y}$$

This leads to $\overline{y} = 1.67a$.

What if the lamina is a shape such as the lamina shown here? There are two ways we can deal with this.



We could split the shape into a rectangle and a square, adding the moment of each shape to determine the total moment.

__+_=_

Alternatively, we could consider the larger square and its moment then subtract the moment of a smaller square to get the moment of the remaining shape.

__ - = __

For this example, we are going to use the second method. In this example it is rather trivial, but for harder examples it is better to use this method. Note that the large square has mass $16a^2\rho$ and the small square has mass $4a^2\rho$, so our lamina has mass $12a^2\rho$.

333

Taking moments about Oy \circlearrowright : $16a^2\rho \times 2a - 4a^2\rho \times 3a = 12a^2\rho \overline{x}$, giving $\overline{x} = \frac{5}{3}a$.

Taking moments about $Ox \circlearrowleft: 16a^2\rho \times 2a - 4a^2\rho \times 3a = 12a^2\rho\overline{y}$, giving $\overline{y} = \frac{5}{3}a$.

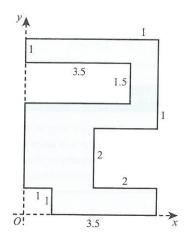
Note that the results for \bar{x} and \bar{y} are the same due to symmetry.

If we look at a more complicated example, such as in the diagram on the right, then we see that a large rectangle, of dimensions 4.5×6.5 , has a 3.5×1.5 rectangle removed, a 2×2 square removed, and a 1×1 square removed.

The mass of the complete rectangle is 29.25ρ . The removed parts are 5.25ρ , 4ρ and ρ , respectively, so the lamina has mass 19ρ . We can now take moments about both axes to find the centre of mass.

About *Oy*
$$\circlearrowleft$$
: $29.25\rho \times 2.25 - 5.25\rho \times 1.75 - 4\rho \times 3.5 - \rho \times 0.5 = 19\rho \overline{x}$, so $\overline{x} = 2.22$.

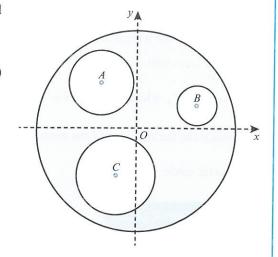
About
$$Ox \circlearrowleft : 29.25\rho \times 3.25 - \rho \times 0.5 - 4\rho \times 2 - 5.25\rho \times 4.75 = 19\rho \overline{y}$$
, so $\overline{y} = 3.24$.



WORKED EXAMPLE 14.9

The diagram shows a large circular-shaped lamina that is constructed by removing smaller circular areas from a circle of radius 15 cm.

The circle centred at the point A(-6, 7) has radius 4cm, the circle centred at B(10, 3) has radius 2cm, and the circle centred at C(-3, -6) has radius 5cm. Relative to the point O, find the centre of mass of the lamina. You may assume the lamina is uniform.



Answer

Large circle mass: 225πρ

Circle A mass: $16\pi\rho$

Circle B mass: $4\pi\rho$

Circle C mass: $25\pi\rho$

So lamina mass is $180\pi\rho$.

Taking moments:

About Oy O:

 $225\pi\rho \times 0 + 25\pi\rho \times 3 + 16\pi\rho \times 6 - 4\pi\rho \times 10 = 180\pi\rho \overline{x}$

So $\bar{x} = 0.728 \, \text{cm}$.

Determine the mass of each smaller circle to obtain the lamina's mass.

Take moments; notice that some contributions are added.

This is actually $-(mass \times (-length))$, so still subtracted but with negative displacement.

About Ox 🖰:

Repeat for \bar{y} .

 $225\pi\rho \times 0 + 25\pi\rho \times 6 - 16\pi\rho \times 7 - 4\pi\rho \times 3 = 180\pi\rho\overline{y}$

So $\bar{y} = 0.144 \, \text{cm}$.

i DID YOU KNOW?

Centres of mass are very important for understanding planetary motion. The centre of mass between two objects is known as the **barycentre**. This is the point at which the two objects balance each other. For example, the barycentre between the Earth and the Moon is offset from the centre of the Earth by approximately 4700 km.

Now we will look at shapes formed from wire. For example, a piece of uniform wire could be bent into an arc of a circle with radius r, and angle 2α radians subtended at the centre.

So each small arc of wire has mass $r\rho d\theta$, and its distance from Oy is

 $r\cos\left(\theta + \frac{\mathrm{d}\theta}{2}\right) \approx r\cos\theta.$

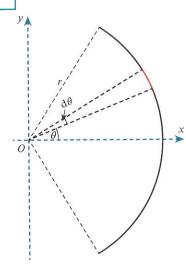
The total mass of the wire is $2r\rho\alpha$, so the moment of the wire about Oy is

$$2r\rho\alpha\overline{x} = \int_{-\alpha}^{\alpha} r^2\rho\cos\theta\,\mathrm{d}\theta.$$

So $2\alpha \overline{x} = r[\sin \theta]_{-\alpha}^{\alpha}$, which leads to $\overline{x} = \frac{r \sin \alpha}{\alpha}$.

For a semicircle, the distance of the centre of mass from O is $\frac{r \sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2r}{\pi}$.

For a quarter circle $\bar{x} = \frac{2\sqrt{2}r}{\pi}$.



WORKED EXAMPLE 14.10

The letter P is constructed using a uniform straight piece of wire of length 4a joined to another piece of uniform wire that is bent into a semicircle of radius a. Given that the semicircular piece of wire is three times as dense as the straight wire, find the position of the centre of mass $G(\overline{x}, \overline{y})$ relative to the point O.

Answer

Mass of straight wire: 4αρ

Mass of semicircle: $\pi a \times 3\rho = 3\pi a\rho$

For
$$\overline{x} \circlearrowleft : 4a\rho \times 0 + 3\pi a\rho \times \frac{2a}{\pi} = (4a\rho + 3\pi a\rho)\overline{x}$$

Hence,
$$\overline{x} = \frac{6a}{4 + 3\pi}$$
.

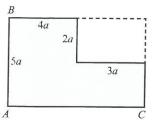
For $\overline{y} \circlearrowleft : 4a\rho \times 2a + 3\pi a\rho \times 3a = (4a\rho + 3\pi a\rho)\overline{y}$

Hence,
$$\bar{y} = \frac{(8+9\pi)a}{4+3\pi}$$
.

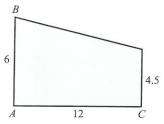
334

EXERCISE 14B

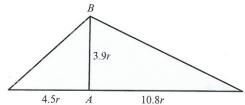
1 The diagram shows a lamina that is formed by removing a small rectangle from a larger rectangle. Find the distance of the centre of mass from AB and from AC.



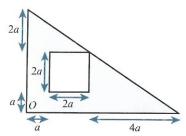
2 The diagram shows a uniform lamina in the shape of a trapezium. Find the distance of the centre of mass from edge AB and from edge AC.



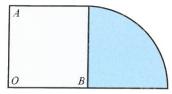
3 The diagram shows two uniform laminas, each a right-angled triangle, that are joined together at one edge, AB. The smaller triangle is twice as dense as the larger triangle. Find the distance of the centre of mass from the edge AB.



4 The image shows a uniform lamina that is formed by removing a square from a right-angled triangle. Find the coordinates of the centre of mass, as measured from the point O.

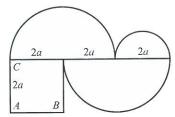


5 The diagram shows a uniform square lamina of side 4r and density 2ρ attached to a uniform lamina in the shape of a quarter circle of radius 4r and density ρ . Find the distance of the centre of mass from the edges OA and OB.

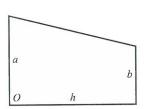


A piece of uniform wire is bent to form the letter D. This letter D consists of a straight edge of length 2m, and a semicircle of radius 1 m. The letter D is held upright with the straight edge in the vertical plane. The bottom corner is denoted as O. Find the centre of mass from the corner O.

- 7 A uniform lamina is made from a square of side 2a joined to a semicircle of radius 2a. This semicircle is then joined to another semicircle of radius 2a, which is in turn joined to a smaller semicircle of radius a, as shown in the diagram. Find the centre of mass from the edges AB and AC.



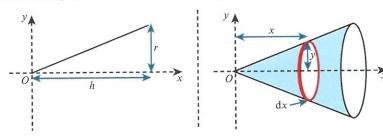
- P
- 8 Show that the centre of mass of this uniform lamina, in the shape of a trapezium, is given by:



$$G\left(\frac{ah + 2bh}{3a + 3b}, \frac{a^2 + ab + b^2}{3a + 3b}\right)$$

14.3 Centres of mass of solids

Moving on from 2-dimensional shapes, we shall now consider 3-dimensional solids. We shall look mainly at the cone and the hemisphere.



Consider rotating the line $y = \frac{r}{h}x$ about the x-axis from x = 0 to x = h, as shown in the diagram. The shape formed is a cone with height h and base radius r. It is symmetrical about the x-axis, and so \overline{y} is zero.

To determine the centre of mass of the solid formed, we must consider a single 'slice' through the cone. A general slice will have radius y, thickness dx, and its volume dV can be written as $\pi y^2 dx$.

Each slice will have mass $\rho \pi y^2 dx$ and is a distance x from the y-axis, so each slice contributes to the moment.

The mass of the cone is then $\frac{1}{3}\rho\pi r^2h$, so $V\overline{x} = \int_0^h \pi xy^2 dx$ will give us the distance of the centre of mass from (0, 0).

So,
$$\frac{1}{3}\rho\pi r^2h\overline{x} = \int_0^h \rho\pi x \left(\frac{r}{h}x\right)^2 dx$$
, which becomes $\frac{1}{3}h\overline{x} = \frac{1}{h^2}\left[\frac{1}{4}x^4\right]_0^h$. Hence, $\overline{x} = \frac{3}{4}h$.

337

() KEY POINT 14.6

The centre of mass of a right circular cone, of height h and base radius r, is given as $G\left(\frac{3}{4}h,0\right)$ when measured from the vertex.

WORKED EXAMPLE 14.11

A solid uniform cone of height 3a and base radius a has a smaller similar cone removed from it. The smaller cone has height a. The resulting shape is known as a **frustum**. Find the centre of mass of the frustum when measured from the smaller of its two plane faces.

Answer

Large cone mass is $\frac{1}{3}\pi a^2 \rho \times 3a = \pi a^3 \rho$.

Small cone mass is $\frac{1}{3}\pi \left(\frac{a}{3}\right)^2 \rho \times a = \frac{1}{27}\pi a^3 \rho$.

So the mass of the frustum is $\frac{26}{27}\pi a^3 \rho$.

From O O:

 $\pi a^3 \rho \times \frac{3}{4} \times 3a - \frac{1}{27} \pi a^3 \rho \times \frac{3}{4} \times a = \frac{26}{27} \pi a^3 \rho \overline{x}$

So $\bar{x} = \frac{30}{13}a$.

So from the smaller plane face $\frac{30}{13}a - a = \frac{17}{13}a$.

First, find the mass of the frustum from the difference in the masses of the two cones.

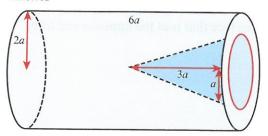
Use the vertex of the larger cone to take moments.

Subtract *a* from this result to get the distance from the smaller plane face.

WORKED EXAMPLE 14.12

A solid uniform cylinder, of radius 2a and length 6a, has a cone of height 3a and base radius a removed from it. The cone removed has its axis of symmetry coinciding with that of the cylinder, and the plane face of the cone lies in the same plane as one end of the cylinder. Find the centre of mass of the remaining solid when measured from the opposite plane face of the cylinder.

Answer



Visualising the solid makes the question much easier.

Mass of cone is $\frac{1}{3}\pi a^2 \times 3a\rho = \pi a^3 \rho$.

Mass of remaining solid is $23\pi a^3 \rho$.

About the opposite face O:

$$24\pi a^3 \rho \times 3a - \pi a^3 \rho \times \left(3a + \frac{9}{4}a\right) = 23\pi a^3 \rho \overline{x}$$

So $\overline{x} = 2.90a$.

Find the mass of each part, subtracting the mass of the cone to get the mass of the remaining solid.

Take moments about the opposite end. Remember the vertex of the cone is a distance of 3a from that face.

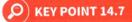
The equation of the circle is $x^2 + y^2 = r^2$.

Each slice is a distance of x from the y-axis, and so each slice is contributing to the moment of the whole hemisphere.

Let the mass of each slice be $\rho \pi y^2 dx$, and let the mass of the hemisphere be $\frac{2}{3}\rho \pi r^3$.

Then, taking moments about Oy for each slice, $\frac{2}{3}\rho\pi r^3\overline{x} = \int_0^r \rho\pi x (r^2 - x^2) dx$, then integrating

gives
$$\frac{2}{3}\pi r^3\overline{x} = \pi \left[\frac{1}{2}r^2x^2 - \frac{1}{4}x^4\right]_0^r$$
, then with the limits, $\frac{2}{3}\pi r^3\overline{x} = \pi \left(\frac{1}{4}r^4\right)$. Hence, $\overline{x} = \frac{3}{8}r$.



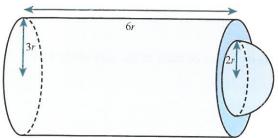
For a solid hemisphere with radius r, the centre of mass is along the line of symmetry $\frac{3}{8}r$ from the centre of the plane face.

EXPLORE 14.1

Consider a solid uniform hemisphere, having centre O and radius r, with a smaller hemisphere, also with centre O but with radius x, removed from it. In groups, investigate what happens to the centre of mass of the remaining body as $x \to r$.

WORKED EXAMPLE 14.13

A solid uniform cylinder of radius 3r and length 6r is connected by one of its plane faces to a solid uniform hemisphere of radius 2r. Their lines of symmetry coincide and the density of the hemisphere is twice that of the cylinder. Find the distance of the centre of mass of the solid from the plane face that is at the opposite end from where the hemisphere is connected.



This diagram shows the situation described in the question.

You could also draw this as separate, smaller diagrams of the cylinder and the hemisphere.

Mass of cylinder: $\pi \times (3r)^2 \times 6r \times \rho = 54\pi r^3 \rho$

Mass of hemisphere: $\frac{2}{3}\pi \times (2r)^3 \times 2\rho = \frac{32}{3}\pi r^3 \rho$

Mass of shape: $\frac{194}{3}\pi r^3 \rho$

About opposite face O:

$$54\pi r^3 \rho \times 3r + \frac{32}{3}\pi r^3 \rho \times \left(6r + \frac{3}{8} \times 2r\right)$$
$$= \frac{194}{3}\pi r^3 \rho \overline{x}$$
So $\overline{x} = \frac{351}{97}r$.

Determine the mass of each part, remembering that the hemisphere has twice the density of the cylinder.

Add the results together.

Take moments about the opposite face to determine \bar{x} .

Consider that you are making a toy that is to be formed by attaching a hemisphere to a cone, as shown in the diagram.

This shape will be placed with the point A on the ground, where A is a point on the rim of the hemisphere. The toy starts with OA vertical. We would like the toy to return to a stable position, that is when the apex (point) of the cone is vertically above O and OA is horizontal. We assume that the cone and the hemisphere are made from the same uniform material.

The mass of the hemisphere is $\frac{2}{3}\pi r^3 \rho$ and the mass of the cone is $\frac{1}{3}\pi r^2 h\rho$.

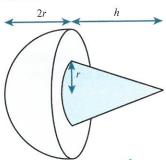
Taking moments about $OA \circlearrowleft: \frac{2}{3}\pi r^3 \rho \times \frac{3}{8}r = \frac{1}{3}\pi r^2 h \rho \times \frac{1}{4}h$

Here we assume that the toy balances when OA is vertical, so we consider h < f(r).

Solving gives $r^2 = \frac{1}{3}h^2$, which simplifies to $h = \sqrt{3}r$. Since we want the toy to return to its upright position, this means that $h < \sqrt{3}r$.



A solid uniform hemisphere of radius 2r is joined to a solid uniform cone of height h and base radius r. The cone and hemisphere have their plane faces joined together. At the join, their lines of symmetry also coincide. If the cone is twice as dense as the hemisphere, find a relationship between h and r so that the shape can balance when both the plane faces are vertical.



Remember that the cone is twice as dense as the hemisphere.

This object has its centre of mass at the join of the two plane faces.

Mass of hemisphere: $\frac{2}{3}\pi \times (2r)^3 \rho = \frac{16}{3}\pi r^3 \rho$

Mass of cone: $\frac{1}{3}\pi r^2 h \times 2\rho = \frac{2}{3}\pi r^2 h\rho$

Moments about the plane face:

 $\frac{16}{3}\pi r^3 \rho \times \frac{3}{8} \times 2r = \frac{2}{3}\pi r^2 h \rho \times \frac{1}{4}h$

So
$$4r^2 = \frac{1}{6}h^2$$
, which means $h = \sqrt{24}r$.

Find the mass of each part. Notice that you do not need the total mass since the moments of the two parts must be equal for equilibrium.

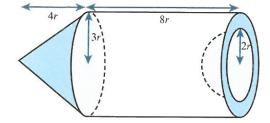
Balance the turning effects and determine h = f(r).

- 1 A uniform solid cylinder, of radius r and length 4r, has a uniform solid hemisphere of radius r, of the same material, attached to one of its plane faces. The plane faces of each solid coincide with each other. Find the distance of the centre of mass from the opposite plane face of the cylinder.
- Two uniform cones with base radius r are joined together by their plane faces. Their lines of symmetry are aligned. The height of one cone is 6r and the height of the other cone is 2r.

 Given that the smaller cone is 50% denser than the larger cone, find the distance of the centre of mass from their joint plane face.
- PS 3 Find, by using integration, the centre of mass of a solid hemisphere of radius 2r, measured from its plane face.
- A uniform solid cylinder has length 4a and radius 2a at each end. Centred on the plane faces are points A and B, respectively, such that AB is 4a. At the plane face B, a hemisphere, of radius 2a and centre B, is removed from the cylinder. Find the centre of mass of the remaining solid from the point A.
- A uniform solid cone, C_1 , of base radius 1.5r and height 4r, is connected to another uniform solid cone, C_2 , of base radius 1.5r and height r. Given that the cones are connected by the faces of their planes, and that C_2 is three times as dense as C_1 , find the centre of mass from the vertex of C_1 .
- A toy is constructed by joining a hemisphere to a cone by their plane faces. Both the hemisphere and cone have the same radius, r, and the cone has height 10r. Given that the density of the cone is ρ , and that the density of the hemisphere is $k\rho$, find the value of k such that the toy can balance when the joint face is vertical.

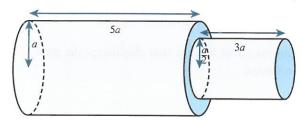


7 The diagram shows a cylinder with a hemisphere removed from one end, and a cone attached to the other end. Each part is solid and its mass uniformly distributed. The density of the cone is twice that of the cylinder. Find the centre of mass from the vertex of the cone.



PS

8 The diagram shows two uniform solid cylinders. The larger cylinder has density ρ and the smaller cylinder has density $k\rho$. The cylinders are joined together by the faces of their planes, and their lines of symmetry coincide. Find the distance of the centre of mass from the plane face that joins the two cylinders.



14.4 Objects in equilibrium

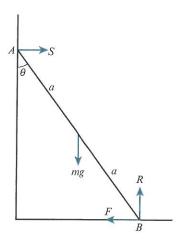
Consider a uniform ladder, AB, resting against a smooth vertical wall and a rough horizontal floor. The ladder has a length of 2a and mass m. The ladder makes an angle of θ with the wall, where $\tan \theta = \frac{3}{4}$.

Can we find the range of values for the coefficient of friction that would keep the ladder from slipping?

First, we resolve forces horizontally to get F = S. Then we resolve forces vertically to get R = mg. If the ladder is to be prevented from slipping, then $F \le \mu R$.

Next, take moments about $B \circlearrowleft : S \times 2a\cos\theta = mg \times a\sin\theta$

With $\sin \theta = \frac{3}{5}$, $\cos \theta = \frac{4}{5}$, we get $S = \frac{3}{8}mg$. Since F = S, $F = \frac{3}{8}mg$, then $\frac{3}{8}mg \le \mu mg$, which leads to $\mu \ge \frac{3}{8}$.



WORKED EXAMPLE 14.15

A uniform ladder is placed against a smooth vertical wall and rough horizontal floor. The ladder is of length a and mass 2m. The ladder is placed such that the angle between the ladder and vertical wall is 30° . A painter, of mass 5m, stands one-quarter of the way up the ladder and the ladder is on the point of slipping. Find the minimum coefficient of friction required to prevent the ladder from slipping.

Make sure you have a fully labelled clear diagram, showing forces and angles.

 $R(\rightarrow)$: S = F and $R(\uparrow)$: R = 7 mg.

Resolve forces in both directions.

About B 🖰:

Take moments about a point that eliminates the most unknown forces.

 $S \times a\cos 30 = 2mg \times \frac{a}{2}\sin 30 + 5mg \times \frac{a}{4}\sin 30$

Hence, $S = \frac{3\sqrt{3}}{4}mg$.

Determine S.

Then, using $F \leqslant \mu R$, $\frac{3\sqrt{3}}{4}mg \leqslant \mu \times 7mg$, Recall that $F = S = \frac{3\sqrt{3}}{4}mg$ and use it to find μ .

so $\mu_{\min} = \frac{3\sqrt{3}}{28}$.

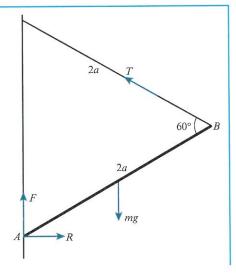
EXPLORE 14.2

Discuss in groups the situations when μ is quite large and close to 1. Can the value of μ be greater than 1? If so, in what situations would this occur? Research online to support your discussions and findings.

WORKED EXAMPLE 14.16

A uniform rod of mass m and length 2a is held in equilibrium by a light, inelastic string of length 2a, and by a frictional force due to the rod's contact with a rough vertical wall. The angle between the string and the rod is 60°, as shown in the diagram. The rod is on the point of slipping downwards.

By resolving forces, and taking moments at an appropriate point, find a range of values for μ so that the rod does slip down.



 $R(\rightarrow)$: $R = T\cos 30$

Resolve forces both horizontally and vertically.

 $R(\uparrow)$: $T\cos 60 + F = mg$

About A \circlearrowleft : $mg \times a \sin 60 = T \cos 30 \times 2a \cos 60 + T \sin 30 \times 2a \sin 60$

Take moments about A.

This gives $T = \frac{1}{2}mg$. So $F = \frac{3}{4}mg$ and $R = \frac{\sqrt{3}}{4}mg$.

Evaluate T.

Then, using $F \leqslant \mu R$, $\frac{3}{4} \leqslant \frac{\mu \sqrt{3}}{4}$ or $\mu \geqslant \sqrt{3}$.

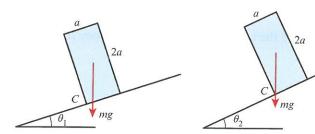
Use limiting friction to establish a range of values for μ .

Since we want the rod to slip, we need $\mu < \sqrt{3}$.

State the correct range.

Consider a rectangular block placed on a rough, sloping plane and imagine that the coefficient of friction is large enough to prevent the block from slipping. The block has dimensions $a \times 2a \times 3a$, where 3a is the depth of the block.

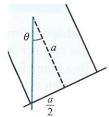
The slope is slowly raised so that the angle increases until the object topples over. We need to find the angle at which the block is about to topple.



When the block is about to topple, its centre of mass will pass through the last point on the edge of the block that contacts the slope. In this case, the centre of mass is vertically above the point C.

Notice that there is a small triangle that contains all the information we require. So when the block is about to topple, the angle θ can be found by considering $\tan \theta = \frac{a}{2a} = \frac{1}{2}$.

So the block topples when $\theta = 26.6^{\circ}$.



A solid uniform hemisphere is placed on a rough slope, with its plane face against the slope. Assume the friction force is great enough to prevent the hemisphere from slipping.

If the slope is inclined at an angle of 65°, state whether or not the hemisphere topples. Justify your answer.

If it does not topple, what is the maximum possible angle of inclination of the slope?

Answer

Let the radius be r.

Quote the standard result for solid uniform hemispheres (see Section 14.3).

Then, using the standard formula for hemispheres, the distance from the slope to the point G is $\frac{3}{8}r$.

Note that the weight passes through the lowest point only when d = r.

Clearly $\tan \theta = \frac{d}{\frac{3}{8}r}$, or $d = \frac{3}{8}r \tan \theta$.

Using $\theta = 65^{\circ}$, $d = \frac{3}{8}r \times 2.145 = 0.804r$, and since

Use the given angle to determine d and show it is less than r.

this is less than r, we can conclude that the hemisphere does not topple.

State that it does not topple.

The maximum angle comes from $r = \frac{3}{8}r \tan \theta$, Find the maximum angle, derived from d = r.

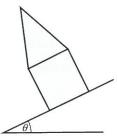
or $\tan \theta = \frac{8}{2}$.

So the maximum angle is 69.4°.

WORKED EXAMPLE 14.18

A shape is formed by joining a solid uniform cylinder to a solid uniform cone. The cylinder has radius r and height r; the cone has base radius r and height 2r. The two solids are joined by a plane face, and the lines of symmetry of the two solids coincide. This shape is placed on a rough slope, as shown in the diagram.

If the slope is sufficiently rough to prevent sliding, find the angle at which the shape is about to topple.



Start with the centre of mass: Mass of cylinder is $\pi r^3 \rho$.

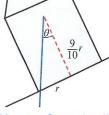
Find the mass of each part, then work out the total

Mass of cone is $\frac{2}{3}\pi r^3 \rho$.

So total mass of solid is $\frac{5}{3}\pi r^3 \rho$.

About base $\circlearrowleft: \pi r^3 \rho \times \frac{r}{2} + \frac{2}{3}\pi r^3 \rho \times \left(r + \frac{r}{2}\right) = \frac{5}{3}\pi r^3 \rho \overline{y}$ So $\overline{y} = \frac{9}{10}r$.

Take moments to find the position of the centre of mass above the slope.



Observe from the diagram that the required angle is θ , and $\tan \theta = \frac{r}{\frac{9}{10}r}$.

Therefore, $\theta = 48.0^{\circ}$.

Next, work out the angle when the shape is about to topple. Use the centre of mass along with the cylinder's radius to determine the angle required.

Take moments about two perpendicular edges to

determine the centre of mass.

Until now, we have been working with objects on a surface. We shall also consider objects that are suspended by a given point.

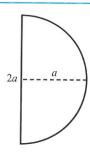
For example, consider a letter D made from uniform wire. We shall model the D as a rod of length 2a, and a semicircle of radius a.

If we hang the letter D from one of its corners, can we find the angle between the rod and the vertical?

We must first find the centre of mass for the D (see Section 14.2). To do this we need to know the masses. The mass of the rod is $2a\rho$, the mass of the curved part is $\pi a\rho$, and the total is $(\pi + 2)a\rho$.

Take moments about the rod $\circlearrowleft: 2a\rho \times 0 + \pi a\rho \times \frac{2a}{\pi} = (\pi + 2)a\rho \overline{x}$, then $\overline{x} = \frac{2a}{\pi + 2}$.

From the lower part of the diagram we can see that $\tan \theta = \frac{\overline{x}}{a}$, which gives $\tan \theta = \frac{2}{\pi + 2}$. So the angle between the rod and the vertical is $\theta = 21.3^{\circ}$.



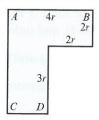


An L-shaped uniform lamina is formed by joining two rectangles together, as shown in the diagram.

a Find the centre of mass of the lamina from the edges AB and AC.

The shape is then suspended from the point A.

b Find the angle between AB and the vertical.



Answer

a Split into $2r \times 5r$ and $2r \times 2r$. Separate the lamina into smaller parts.

About AC O:

$$10r^2\rho \times r + 4r^2\rho \times 3r = 14r^2\rho \overline{x}$$
, so $\overline{x} = \frac{11}{7}r$.

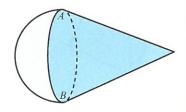
About AB 🖰:

$$10r^2\rho \times \frac{5}{2}r + 4r^2\rho \times r = 14r^2\rho \overline{y}$$
, so $\overline{y} = \frac{29}{14}r$.

b Using $\tan \theta = \frac{\overline{y}}{\overline{x}}$ leads to $\tan \theta = \frac{29}{22}$. Write down the tangent of the angle in terms of \overline{x} , \overline{y} . And so $\theta = 52.8^{\circ}$.

WORKED EXAMPLE 14.20

The diagram shows a solid uniform cone joined to a solid uniform hemisphere. The cone has base radius r and height 3r, and the hemisphere has radius r. The two shapes are joined by their plane faces, and AB is a diameter on that plane face.



If the density of the cone is four times that of the hemisphere, find the position of the centre of mass relative to the line AB. The shape is then suspended from point A. Find the angle AB makes with the vertical.

Answer

Mass of cone is
$$\frac{1}{3}\pi r^2 \times 3r \times 4\rho = 4\pi r^3 \rho$$
.

Determine the mass of each part first.

Mass of hemisphere is $\frac{2}{3}\pi r^3 \rho$.

Total mass is
$$\frac{14}{3}\pi r^3 \rho$$
.

Total mass for the whole solid.

About AB O:

Take moments about the diameter AB.

$$4\pi r^3 \rho \times \frac{3}{4}r - \frac{2}{3}\pi r^3 \rho \times \frac{3}{8}r = \frac{14}{3}\pi r^3 \rho \overline{x}$$

Simplify:
$$\overline{x} = \frac{33}{56}r$$

Obtain \bar{x} .

For the angle,
$$\tan \theta = \frac{\frac{33}{56}r}{\frac{56}{r}}$$
, which gives $\theta = 30.5^{\circ}$.

Use this in a triangle with adjacent side equal to the radius r.

Lastly, we shall look at objects sliding versus toppling. Consider a cuboid of dimensions $2a \times 2a \times 4a$ and mass m resting on rough, horizontal ground. We are going to apply a force X at the top edge of the cuboid. This force will either make the cuboid slide along the ground or make it topple about O.

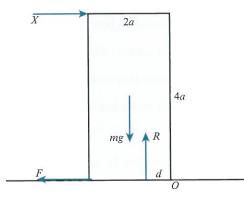
As the force X increases, the reaction force gets closer and closer to the point O, and unless the cuboid slides it will topple over.

So resolving and taking moments, $R(\rightarrow)$: X = F, $R(\uparrow)$: R = mg.

Then taking moments about $O \circlearrowleft : X \times 4a + R \times d = mg \times a$

So
$$X = \frac{mga - mgd}{4a}$$
.

If X = 0, d = a, so the reaction force is midway along the edge that touches the ground.



If
$$X = \frac{1}{4}mg$$
 then $mga = mga - mgd$, which gives $d = 0$. This means

the cuboid is on the point of toppling. Now we consider the possibility of sliding. If

 $X = F = \mu R$, then if $\mu > \frac{1}{4}$, the cuboid will topple before sliding.

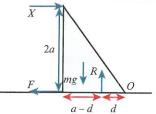
If
$$X = \frac{1}{6}mg$$
 and $\mu = \frac{1}{8}$, we note that $X = \frac{mga - mgd}{4a}$ gives $d = \frac{1}{3}a$, which means the shape

will not topple. Then, noting that $F_{\text{max}} = \frac{1}{8}mg$ means that X > F, and we can see that the cuboid slides.

WORKED EXAMPLE 14.21

A uniform right-angled triangular prism, of mass m, is resting on a rough horizontal surface, as shown in the diagram. The triangle has sides a, 2a, $\sqrt{5}a$ and the depth of the prism is a. A force, X, is applied to the top edge, as shown.

Determine in each case whether or not the prism breaks equilibrium. If it does, determine if it slides or topples.



a
$$X = \frac{1}{5}mg, \mu = \frac{3}{4}$$

b
$$X = \frac{1}{4}mg, \mu = \frac{1}{5}$$

c
$$X = \frac{1}{3}mg, \mu = \frac{1}{2}$$

Answer

a Resolving $R(\rightarrow)$: X = F and $R(\uparrow)$: R = mg. Always resolve forces, then take moments to set up your system.

Moments about $O \circlearrowleft: X \times 2a + R \times d = mg \times \frac{2}{3}a$

So
$$X = \frac{\frac{2}{3}mga - mgd}{2a}$$
.

So $X = \frac{1}{5}mg \Rightarrow d = \frac{4}{15}a$ so it will not topple. d > 0 so it will not topple.

Using $F = \mu R$ we get $F = \frac{3}{4}mg > \frac{1}{5}mg$ so it F > X so it will not slide.

Equilibrium is not broken.

b With $X = \frac{1}{4}mg \Rightarrow d = \frac{1}{6}a$ so it will not topple. d > 0 so it will not topple.

 $F = \frac{1}{5}mg < \frac{1}{4}mg$ so the prism breaks F < X so sliding occurs. equilibrium by sliding.

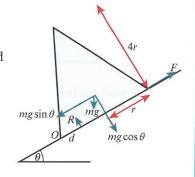
c With $X = \frac{1}{3}mg \Rightarrow d = 0$ so the prism is d = 0 so toppling occurs. about to topple.

 $F = \frac{1}{2}mg > \frac{1}{3}mg$ so the prism topples F > X so it will not slide. but doesn't slide.

WORKED EXAMPLE 14.22

A solid uniform cone, of base radius r and height 4r, is placed on a rough plane inclined at an angle θ , as shown in the diagram.

The coefficient of friction between the cone and the plane is 0.5. The plane is hinged at the bottom, and it is slowly rotated so that θ increases. Giving a justification for your answer, determine whether or not the cone topples before it slides.



Resolving $R(\nearrow)$: $F = mg \sin \theta$ and $R(\nwarrow)$: $R = mg \cos \theta$.

Resolve forces parallel to the plane and perpendicular to the plane.

Taking moments about $O
ightharpoonup : R \times d + mg \sin \theta \times \frac{1}{4} \times 4r$

Split mg into components and take moments about the bottom point of contact, O.

 $= mg\cos\theta \times r$

Rearranging gives $d = \frac{mgr\cos\theta - mgr\sin\theta}{mg\cos\theta}$.

So if d = 0 then $\cos \theta = \sin \theta \Rightarrow \theta = 45^{\circ}$. So the cone will be on the point of toppling if this angle is reached. Determine the limit for toppling.

Assuming the cone is on the point of sliding, $mg \sin \theta = \mu mg \cos \theta$, hence $\mu = \tan \theta$.

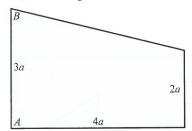
Use the resolved results to find the sliding limit.

Since $\mu = 0.5$, $\theta = 26.6$.

So the cone will slide before it topples, when θ is 26.6°. State how equilibrium is broken.

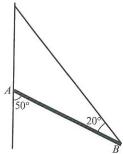
EXERCISE 14D

- 1 A uniform solid cylinder, of radius r and height 5r, is suspended from a point on the rim of its plane face. It is allowed to rest in equilibrium. Find the angle between the plane face of the cylinder and the downwards vertical.
- 2 A uniform solid cylinder, of radius 2r and height 7r, is resting on a sufficiently rough slope. The slope is inclined at an angle α . Find the maximum value of α such that the cylinder is on the point of toppling.
- 3 A ladder of length 4a is placed such that it rests against a smooth vertical wall and stands upon a rough horizontal floor. The angle between the ladder and the wall is 30°. The ladder has mass 2m. Find the range of values of the coefficient of friction, μ , so that the ladder does not slip.
- A solid uniform cone, of base radius 2a and height 5a, is suspended by a point, B, on the rim of its circular base. The centre of the circular base is denoted by C. Find the angle BC makes with the vertical.
 - 5 The diagram shows a uniform lamina in the shape of a trapezium. The lamina is suspended from the point A. Find the angle between the vertical and the edge AB.



- 6 A uniform ladder, of length 2a and mass m, is resting against a smooth vertical wall and a rough horizontal floor. The ladder is making an angle of 30° with the wall, and the coefficient of friction between the ladder and the floor is $\frac{1}{2\sqrt{3}}$. An electrician, of mass 8m, is trying to ascend the ladder. Determine how far they can walk up the ladder before it slips.

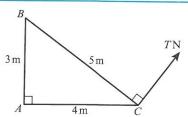
- M
- 7 The diagram shows a uniform rod, of mass m and length 2a, smoothly hinged to a vertical wall. A light, inelastic string connects the rod to a point on the wall above the hinge. Find the magnitude and direction of the force on the rod from the hinge.



- PS
- A solid uniform cone, of base radius r and height 6r, has a similar smaller cone of height 2r removed from the top to form a frustum. This frustum is placed, with its larger plane face, on a rough surface that is hinged to the floor at one edge. The surface is slowly rotated so that the incline angle increases. Given that the coefficient of friction is 0.85, find whether the frustum breaks equilibrium by toppling or sliding.
- PS
- A solid uniform hemisphere, of radius r, is placed onto a rough plane inclined at 45° to the horizontal. A force, P, parallel to and up the plane, is applied to the hemisphere at a point that is $\frac{r}{2}$ above the surface of the plane. The highest point on the rim of the hemisphere that touches the plane is denoted by A.
 - a Assuming that the friction is great enough to prevent slipping, find the value of P required to make the hemisphere topple up the plane. Give your answer in terms of m and g.
 - b Let $\mu = \frac{3}{4}$ and let the hemisphere be on the point of slipping up the plane. Find the distance between the reaction force and the point A.

WORKED PAST PAPER QUESTION

Uniform rods AB, AC and BC have lengths 3 m, 4 m and 5 m respectively, and weights 15 N, 20 N and 25 N respectively. The rods are rigidly joined to form a right-angled triangular frame ABC. The frame is hinged at B to a fixed point and is held in equilibrium, with AC horizontal, by means of an inextensible string attached at C. The string is at right angles to BC and the tension in the string is T N (see diagram).



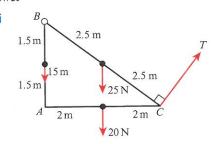
i Find the value of T.

A uniform triangular lamina PQR, of weight 60 N, has the same size and shape as the frame ABC. The lamina is now attached to the frame with P, Q and R at A, B and C respectively. The composite body is held in equilibrium with A, B and C in the same positions as before. Find

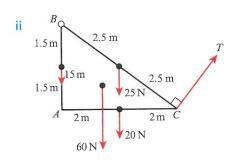
- ii the new value of T.
- iii the magnitude of the vertical component of the force acting on the composite body at B.

Cambridge International AS & A Level Mathematics 9709 Paper 5 Q4 June 2008





Moments about $B \circlearrowleft : 20 \times 2 + 25 \times 2 = 5T$. Hence, T = 18 N.



Moments about $B \circlearrowleft : 20 \times 2 + 25 \times 2 + 60 \times \frac{1}{3} \times 4 = 5T$. Hence, T = 34 N.

iii Let the force be Y, then $R(\uparrow)$: $Y + T \times \frac{4}{5} = 15 + 20 + 25 + 60$. Hence, Y = 92.8 N.

Checklist of learning and understanding

Moment of a force:

• A force F with perpendicular distance d from a point O has moment Fd about the point O.

Centres of mass, 2D lamina and frameworks:

- For a composite body made up of masses m_1, m_2, \dots with distances x_1, x_2, \dots , from a reference point respectively, the distance of the centre of mass from the reference point is $\overline{x} = \frac{\sum x_i m_i}{\sum m_i}.$
- For a triangular lamina the centre of mass is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, where (x_1, y_1) , (x_2, y_2) and (x_3, x_3) are the coordinates of the vertices of the triangle.
- For a lamina of a sector of angle 2α radians from a circle centre O, with radius r, the centre of mass is $\frac{2r\sin\alpha}{3\alpha}$ from the centre of the circle.
- For an arc of a circle of angle 2α radians from a circle centre O, with radius r, the centre of mass is $\frac{r \sin \alpha}{\alpha}$ from the centre of the circle.

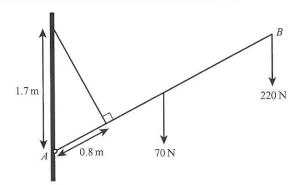
Centres of mass, 3D solids:

- For a right circular cone, with height h, the centre of mass is along the line of symmetry through the vertex, at a distance $\frac{3}{4}h$ from the vertex of the cone.
- For a solid hemisphere with radius r, the centre of mass is along the line of symmetry $\frac{3}{8}r$ from the centre of the plane face.

END-OF-CHAPTER REVIEW EXERCISE 14



A uniform beam AB has length 2m and weight 70 N. The beam is hinged at A to a fixed point on a vertical wall, and is held in equilibrium by a light inextensible rope. One end of the rope is attached to the wall at a point 1.7m vertically above the hinge. The other end of the rope is attached to the beam at a point 0.8m from A. The rope is at right angles to AB. The beam carries a load of weight 220 N at B (see diagram).

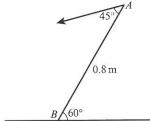


- i Find the tension in the rope.
- ii Find the direction of the force exerted on the beam at A.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q4 November 2010



A uniform rod AB has weight 6N and length 0.8 m. The rod rests in limiting equilibrium with B in contact with a rough horizontal surface and AB inclined at 60° to the horizontal. Equilibrium is maintained by a force, in the vertical plane containing AB, acting at A at an angle of 45° to AB (see diagram). Calculate

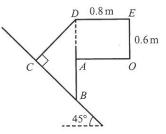


- i the magnitude of the force applied at A,
- ii the least possible value of the coefficient of friction at B.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q2 November 2012



The diagram shows the cross-section OABCDE through the centre of mass of a uniform prism on a rough inclined plane. The portion ADEO is a rectangle in which AD = OE = 0.6 m and DE = AO = 0.8 m; the portion BCD is an isosceles triangle in which angle BCD is a right angle, and A is the mid-point of BD. The plane is inclined at 45° to the horizontal, BC lies along a line of greatest slope of the plane and DE is horizontal.



i Calculate the distance of the centre of mass of the prism from BD.

The weight of the prism is 21 N, and it is held in equilibrium by a horizontal force of magnitude P N acting along ED.

- ii a Find the smallest value of P for which the prism does not topple.
 - **b** It is given that the prism is about to slip for this smallest value of *P*. Calculate the coefficient of friction between the prism and the plane.

The value of P is gradually increased until the prism ceases to be in equilibrium.

iii Show that the prism topples before it begins to slide, stating the value of P at which equilibrium is broken.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q7 June 2015

Chapter 15 Circular motion

In this chapter you will learn how to:

- relate angular speed (measured in rad s⁻¹) and linear speed
- apply the formula $r\omega^2$ or $\frac{v^2}{r}$ (for the acceleration towards the centre of a circle) to problems involving horizontal circles and conical pendulums with constant angular speed and to problems involving vertical circles without any loss in energy.

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Mechanics, Chapter 3	Resolve forces in two perpendicular directions.	Find the magnitude and direction of the resultant of the forces shown. 7N y 5N 43° 50° x
AS & A Level Mathematics Mechanics, Chapters 2 & 3	Apply Newton's second law with multiple forces.	2 A block on rough horizontal ground is subject to three forces: a pulling force of magnitude 17.8 N acting at 20° above the horizontal; a horizontal pushing force, in the same direction as the pulling force, of magnitude 8.9 N; and a resisting force of magnitude 22.3 N. If the block has mass 4 kg, find the acceleration of the block.
AS & A Level Mathematics Mechanics, Chapters 8 & 9	Calculate the potential energy and kinetic energy of a system at any point during its motion.	3 A particle of mass 1.5 kg is projected up a smooth slope inclined at 25°. The particle has initial speed 22 m s ⁻¹ . If the particle travels 5 m up the slope, what is its speed at that point?

What is circular motion?

Circular motion is the motion of a particle around part or all of a circular path. The circle may be horizontal or vertical.

Examples of circular motion are all around us in real life, but some are more obvious than others. The pendulum on an old clock and a conker on a string both move with circular motion, but have you ever thought of cars racing in the Indianapolis 500 in the same way? Because the track is banked, the cars are moving in the same way as a steel ball around the inside of a circular bearing.

In this chapter we shall look at two types of circular motion: horizontal and vertical. We shall discover that the acceleration of a particle moving in a circle is directed towards the centre of the circle. We shall often use **conservation of energy** and Newton's second law to solve problems of circular motion.

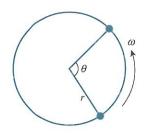
In this chapter we shall use the symbols ν for linear speed, ω for angular speed, r for radius, and α for acceleration. We shall use the abbreviations KE for kinetic energy and PE for potential energy.

Unless stated otherwise, $g = 10 \,\mathrm{m \, s^{-2}}$.

15.1 Horizontal circles

Consider a particle moving in a horizontal circular path with constant speed.

The path is an arc of a circle, measured as $r\theta$. The rate of change of this distance $\frac{\mathrm{d}}{\mathrm{d}t}(r\theta)$ is equal to $r\frac{\mathrm{d}\theta}{\mathrm{d}t}$. So we have the relation $v=r\frac{\mathrm{d}\theta}{\mathrm{d}t}$ or $v=r\omega$, as shown in Key point 15.1. The quantity $\omega=\frac{\mathrm{d}\theta}{\mathrm{d}t}$ is known as the **angular speed** and is constant.



(D) KEY POINT 15.1

To calculate the speed of a particle moving with circular motion, use:

$$v = r \frac{\mathrm{d}\theta}{\mathrm{d}t}$$
 or $v = r\omega$

The formulae in Key point 15.1 imply that, the larger the circle, the greater the speed must be. The angular speed is the same whatever the size of the radius because it measures the angle of travel over time, as shown in Key point 15.2.

(A) KEY POINT 15.2

The angular speed ω is measured in radians per second, denoted as rad s⁻¹. One complete cycle is 2π radians, so the frequency, the number of revolutions per second, is $\frac{2\pi}{\omega}$.

WORKED EXAMPLE 15.1

In each case, convert either angular speed to linear speed or linear speed to angular speed, or calculate both quantities, as appropriate:

- a a particle travelling in a circular path of radius 5 m with angular speed 4 rad s⁻¹
- b a particle travelling along a circular path of radius 2.5 m, with speed $4\,\mathrm{m\,s^{-1}}$
- c a particle travelling on a circular path with radius 2m, where the time to cover one complete circle is 6s
- d the Earth travelling around the Sun, where the path is assumed to be circular and of radius $1.496 \times 10^{11} \, \mathrm{m}$.

Answer

- a Using $v = r\omega$, $v = 5 \times 4 = 20 \,\mathrm{m \, s^{-1}}$. The first two examples just require you to substitute values into the formula.
- **b** Using $v = r\omega$, $4 = 2.5\omega$. Hence, $\omega = 1.6 \,\mathrm{rad}\,\mathrm{s}^{-1}$.
- c Since 2π is covered in 6s, $\omega = \frac{2\pi}{6} = \frac{\pi}{3} \operatorname{rad} \operatorname{s}^{-1}$. First determine the angular speed.
 - Then $v = 2 \times \frac{\pi}{3} = \frac{2\pi}{3} \,\text{m s}^{-1}$. Then find the linear speed.
- d First, consider that 365 days is 31 536 000 seconds.

 Determine the number of seconds per year.

$$\omega = \frac{2\pi}{31\,536\,000}$$

\$\approx 1.99 \times 10^{-7} \text{ rad s}^{-1}\$

Use $v = r\omega$:

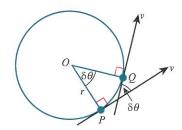
Calculate the linear speed.

$$v = 1.496 \times 10^{11} \times 1.992 \times 10^{-7}$$

= 29 800 m s⁻¹

As an object travels in a circular path, its direction of motion constantly changes.

Because of this, the linear speed of the object is not actually constant. Consider two points on the path of a circle, P and Q. Let the angle POQ be very small, $\delta\theta$. At the point P we assume the speed is v. Then, since Q is very close to P, we can also assume that the speed at Q has the components $v\cos\delta\theta$ and $v\sin\delta\theta$, where these components are parallel and perpendicular to the tangent at P. Since $\sin\delta\theta \approx \delta\theta$ and $\cos\delta\theta \approx 1$, we can focus on the acceleration parallel and perpendicular to the tangent at P.



Since acceleration = $\frac{\text{change in speed}}{\text{change in time}}$, the parallel acceleration component is $\frac{v\cos\delta\theta-v}{\delta t}\approx 0$, and the perpendicular acceleration component, acting towards the centre, is $\frac{v\sin\delta\theta-v}{\delta t}$.

Hence, the acceleration acts towards the centre and is of magnitude $v \frac{d\theta}{dt}$. This is commonly written as $r\omega^2$ or $\frac{v^2}{r}$, as shown in Key point 15.3.

(D) KEY POINT 15.3

The acceleration towards the centre for an object moving in circular motion is $a = \frac{v^2}{r}$ or $a = r\omega^2$. The link between v and ω is $v = r\omega$.

Consider a particle travelling on a circular path of radius 1.2m. Its speed is 3 m s⁻¹. If the mass of the particle is 2 kg, can we find the force towards the centre?

We begin with $F = ma = m \frac{v^2}{r}$.

Substituting the mass, radius and speed provided: $F = 2 \times \frac{3^2}{1.2}$

Hence, $F = 15 \,\mathrm{N}$.

WORKED EXAMPLE 15.2

In each case, find the force towards the centre:

- a a particle of mass $1.5 \,\mathrm{kg}$, travelling with angular speed $4 \,\mathrm{rad}\,\mathrm{s}^{-1}$, where the radius is $0.5 \,\mathrm{m}$
- b a particle of mass 4 kg, travelling with speed $5\,\mathrm{m\,s^{-1}}$, where the radius is $2.4\,\mathrm{m}$
- c a particle of mass 0.5 kg, travelling around a circle of radius 20 m in 12 s.

357

Answer

- a Using F = ma, we have $F = mr\omega^2$. Note the type of speed and use the corresponding formula.
- **b** Using $F = m \frac{v^2}{r}$, we have $4 \times \frac{5^2}{2.4}$. So $F = \frac{125}{3}$ N.
- c First find $\omega = \frac{2\pi}{12} = \frac{\pi}{6}$. Determine the angular speed before applying F = ma.

Then using $F = mr\omega^2$ gives $0.5 \times 20 \times \left(\frac{\pi}{6}\right)^2 = \frac{5\pi^2}{18} = 2.74 \text{ N}.$

Now that we understand the idea of acceleration towards the centre of the circle, let us look at an example in context. Consider a particle that is attached to one end of a light, inextensible string of length 2 m. The other end of the string is attached to a point on a smooth, horizontal table. The particle has mass 1.5 kg and it describes horizontal circles on the surface of the table with angular speed 4 rad s⁻¹. We want to find the tension in the string.

Start with $F = ma = mr\omega^2$, then $T = 1.5 \times 2 \times 4^2$, so the tension in the string is 48 N.

WORKED EXAMPLE 15.3

A particle is describing circles on a smooth, horizontal table. The particle is attached to a light, inelastic string that is attached to a point, O, where the radius of the circles is 1.2 m. Given that the tension in the string is 40 N, and that the particle can complete one circle in $\frac{5\pi}{3}$ s, find the mass of the particle.

Answer

First use $\omega = \frac{2\pi}{\frac{3\pi}{5}} = 1.2 \text{ rad s}^{-1}$. Find the angular speed.

Then F = ma gives $40 = m \times 1.2 \times (1.2)^2$. Apply F = ma.

So $m = 23.1 \, \text{kg}$.

WORKED EXAMPLE 15.4

A particle is placed on a rough horizontal disc $4\,\mathrm{m}$ from the centre of the disc. The particle has mass $2\,\mathrm{kg}$ and the coefficient of friction between the particle and the disc is 0.6. The disc begins to spin around slowly until the particle slips. Find the frictional force on the particle when the disc is spinning at $2\,\mathrm{m\,s^{-1}}$. Find also the angular speed of the disc when the particle is on the point of slipping.

Draw a fully labelled, clear diagram.

This helps you to see what is happening.

Using F = ma, we have

$$F = 2 \times \frac{2^2}{4} = 2 \,\mathrm{N}$$

Use $F = m \frac{v^2}{r}$ to obtain the result.

At the slipping point,

$$F_{\text{lim}} = \mu R = 0.6 \times 2g = 12 \text{ N}.$$

State the maximum frictional force (see Rewind).

So $12 = 2 \times 4 \times \omega^2$.

Hence,
$$\omega = \frac{\sqrt{6}}{2} \operatorname{rad} s^{-1}$$
.

(see Rewind).

Use $F = mr\omega^2$ to find the angular speed.

Recall from Chapter 14 that when an object is on the point of

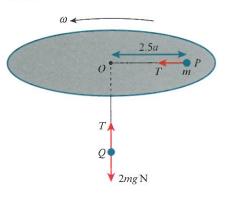
slipping on a rough surface, $F_{\text{max}} = \mu R$. We can also write F_{max}

as F_{lim} , which refers to limiting friction.

WORKED EXAMPLE 15.5

Two particles are attached to a light, inelastic string of length 4a. One particle, P, of mass m, is placed on a large, smooth disc. The string is then threaded through a smooth hole in the centre of the disc. At the other end of the string is the particle Q, of mass 2m. The disc is spinning such that particle Q does not move and is 1.5a below the centre of the disc. Find the tension in the string, and also find the angular speed of the particle P in terms of A as it moves in circles.

Answer



Since the string is of length 4a it is clear that the radius of the circle described is 2.5a.

Q does not move, so T = 2mg N

Balance forces vertically for Q.

So $T = mr\omega^2$ gives $2mg = m \times 2.5a \times \omega^2$

Use F = ma with angular speed form for particle Q.

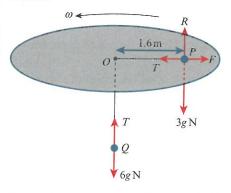
Hence, $\omega = \sqrt{\frac{4g}{5a}} \text{rad s}^{-1}$.

Find ω .

WORKED EXAMPLE 15.6

A particle P, is placed on a large, rough, horizontal disc. P is of mass 3 kg. P is then attached to a light, inextensible string of length 2.4 m. The string passes through a smooth hole in the centre of the disc, and at the other end is a particle, Q, of mass 6 kg. The coefficient of friction between P and the disc is 0.5. Q is hanging freely under the hole in the disc. The disc begins to spin with P at a distance of 1.6 m from the centre of the disc. If Q does not move at any point during the motion, find the range of angular speeds possible.

Answer



The friction force will be used in conjunction with the force towards the centre.

First, $F_{\text{lim}} = 0.5 \times 3g = 15 \,\text{N}$, and if Q doesn't move $T = 60 \,\text{N}$. Determine the limiting friction and state the

tension in the string.

Slipping inwards: $60 = F + mr\omega^2$ Then $60 = 15 + 3 \times 1.6\omega^2$

For slipping inwards, ω has a minimum value.

 $\omega = \frac{5\sqrt{6}}{4} \text{rad s}^{-1}$ Slipping outwards: $60 + F = mr\omega^2$

For slipping outwards, ω has a maximum Then $75 = 4.8 \omega^2$ value.

$$\omega = \frac{5\sqrt{10}}{4} \text{rad s}^{-1}$$

Hence, $\frac{5\sqrt{6}}{4} \leqslant \omega \leqslant \frac{5\sqrt{10}}{4}$

State the range of allowed values.

DID YOU KNOW?

Rollercoaster loops are never circular. They are actually constructed to form what is known as a clothoid loop. On a circular track, the speed would decrease as you travelled around the loop with constant acceleration and the roller coaster may not complete the loop. On a clothoid loop, the radius is smaller at the top to keep the speed high enough.

EXERCISE 15A

- 1 In each of the following, use the relation $v = r\omega$ to determine the unknown value.
 - a v = 4, r = 6. Find ω .
 - **b** $v = 6, \omega = 3$. Find r.
 - c $r = 5, \omega = 0.8$. Find v.

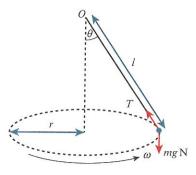
- 2 A particle of mass 0.2 kg is attached to a light inextensible string of length 1.5 m. The particle is moving in horizontal circles. Find the tension in the string if the speed of the particle is 5 m s⁻¹.
- 3 A particle of mass 0.8 kg is attached to a light inextensible string of length 1.2 m. The particle is moving in horizontal circles. Find the angular speed of the particle if the tension in the string is 10 N.
- 4 A particle is describing horizontal circles on a smooth, horizontal table. The particle is fixed to its circular path by an inelastic string of length 1.5 m. Given that the tension in the string is 45 N, and that the mass of the particle is 5 kg, find the angular speed of the particle.

State also how many seconds the particle takes to describe one complete circle.

- 5 A particle is placed on a rough horizontal disc, 2a from the centre of the disc. The particle is of mass 3m. The disc then starts to spin at an angular speed of $\sqrt{\frac{g}{4a}}$. Given that the particle is on the point of slipping, find a range of values of μ such that equilibrium is not broken.
- 6 A car is driving on a horizontal circular section of road that has radius 50 m. The car is of mass 800 kg and the coefficient of friction between the tyres and the road is 0.8. Find the maximum speed that the car can drive around the road without slipping.
- A particle of mass 1.5 kg is resting on a smooth horizontal table. The particle is attached to a light, inextensible string of length 2.5 m. This string is passed through a smooth hole in the table. At the other end of the string is a particle of mass 4 kg. The particle on the table is set in motion and describes circles with radius 1.5 m. Find the speed of the particle on the table. Assume that the particle that is freely hanging does not move during the motion.
 - A particle is describing horizontal circles of radius a; the mass of the particle is m. Given that the particle can complete one circle in τ seconds, and that the particle is held in its path by means of a light, inextensible string with tension T, find an expression for the mass m.
 - A particle is placed on the inside of a rough, hollow cylinder of radius 2a. The cylinder is placed upright and can rotate about an axis through its centre. The particle has mass m and the coefficient of friction is $\frac{3}{4}$. Given the particle does not slip as the cylinder rotates, find the speed at which the cylinder rotates.
- 10 A particle, P, is placed on a large, rough, horizontal disc. P is of mass 2.4 kg. The particle is then attached to a light, inextensible string of length 4 m. The string passes through a smooth hole in the centre of the disc, and at the other end is a particle, Q, of mass 3 kg. The coefficient of friction between P and the disc is 0.2. Q is hanging freely under the hole in the disc. The disc is set spinning with P at a distance of 1.2 m from the centre of the disc. If Q does not move at any point during the motion, find the range of speeds possible.

15.2 The 3-dimensional case

Consider a particle on the end of a light, inelastic string with the other end of the string attached to a fixed point O. The particle is set in motion to describe small horizontal circles. This is a **conical pendulum** model.



The string describes the curved surface of a cone as the particle moves in a circle below the fixed point at O. The force towards the centre is now a component of the tension in the string. The vertical component of the tension links to the weight of the particle. With this kind of problem we must resolve into horizontal and vertical components first.

$$R(\uparrow)$$
: $T\cos\theta = mg$; hence, $T = \frac{mg}{\cos\theta}$.

Then $R(\leftarrow)$: $T\sin\theta = mr\omega^2$.

Since $\frac{r}{l} = \sin \theta$, it follows that $T \sin \theta = ml \sin \theta \omega^2$, or $T = ml\omega^2$.

Equating these two results gives $\omega = \sqrt{\frac{g}{l\cos\theta}}$.

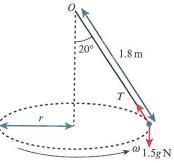
Assigning some values, let the mass of the particle be 2kg, the angle 30° and the string length 2.5 m.

Then the tension is $\frac{40\sqrt{3}}{3}N$ and the angular speed is $\sqrt{\frac{8\sqrt{3}}{3}}rads^{-1}$.

WORKED EXAMPLE 15.7

A particle of mass 1.5 kg is attached to one end of a light, inelastic string of length 1.8 m. The string is attached to a fixed point, O, at its other end. The particle is set in motion and describes horizontal circles with the string taut and at 20° to the downward vertical through the point O. Find the tension in the string and the angular speed of the particle.

Answer



A large, well-labelled diagram is essential for mechanics problems.

Draw a diagram so you can resolve both vertically and horizontally.

 $R(\uparrow)$: $T\cos 20 = 1.5g$; hence, $T = \frac{15}{\cos 20} = 16.0 \text{ N}$. Resolve vertically to find T.

 $R(\leftarrow)$: $T\sin 20 = 1.5 \times r \times \omega^2$, where $r = 1.8\sin 20$. Resolve horizontally using F = ma

 $\frac{15}{\cos 20}\sin 20 = 1.5 \times 1.8\sin 20 \times \omega^2$ Solve to find ω .

So $\omega = 2.43 \, \text{rad s}^{-1}$.

362

If you are asked to find the speed of a particle instead of the angular speed, you need to be careful when cancelling terms.

For the general case in the diagram at the start of Section 15.2, $T = \frac{mg}{\cos \theta}$ is unchanged,

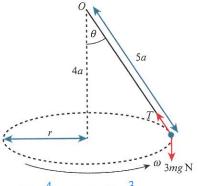
but towards the centre, $T \sin \theta = m \frac{v^2}{l \sin \theta}$.

Notice that the $\sin \theta$ term does not cancel here. Keep in mind this term cancels only for angular speed.

WORKED EXAMPLE 15.8

A particle of mass 3m is attached to a light, inextensible string of length 5a. The other end of the string is attached to a fixed point, O. The particle is moving in horizontal circles about a centre that is 4a vertically below the point O. By resolving vertically and horizontally, find the tension in the string and the speed of the particle.

Answer



$$\cos \theta = \frac{4}{5}$$
 and $\sin \theta = \frac{3}{5}$.

 $R(\uparrow)$: $T\cos\theta = 3mg$, and so $T = \frac{15}{4}mg$.

$$R(\leftarrow)$$
: $T\sin\theta = 3m\frac{v^2}{3a}$, then $\frac{15}{4}mg \times \frac{3}{5} = \frac{m}{a}v^2$.

Hence, $v = \frac{3}{2}\sqrt{ga}$.

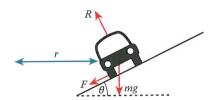
First note the sides of the triangle formed.

Resolve vertically to find the tension.

Resolve horizontally with this tension to get v^2 and substitute the earlier expression found for T.

State the value of v.

Consider an object on a sloped surface, sometimes called a banked surface. Here, there is a reaction force rather than tension in a string.



First consider the case when the surface is smooth. We shall have the same situation as that of the conical pendulum. When there is friction, it is usually considered to act down the slope. This is because the car is likely to slip up the slope due to excess speed.

Resolving vertically: $R\cos\theta = mg + F\sin\theta$

Then considering the forces towards the centre: $F\cos\theta + R\sin\theta = m\frac{v^2}{r}$

If the car is on the point of slipping, $F = \mu R$.

Resolving vertically,
$$R\cos\theta - \mu R\sin\theta = mg$$
, so $R = \frac{mg}{\cos\theta - \mu\sin\theta}$

Then from the force towards the centre, $R(\mu\cos\theta + \sin\theta) = m\frac{v^2}{r}$, so $\frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta}mg = m\frac{v^2}{r}$.

If we then divide top and bottom by $\cos \theta$, this gives $\frac{v^2}{r} = g \frac{\mu + \tan \theta}{1 - \mu \tan \theta}$.

You do not have to memorise this result, but you should be able to derive it.

WORKED EXAMPLE 15.9

A car of mass $1000 \,\mathrm{kg}$ is turning on a banked road of radius $80 \,\mathrm{m}$. The incline of the road is 15° and the car is travelling at $25 \,\mathrm{m\,s^{-1}}$. The car is about to slip up the road.

- a Resolve vertically and find an expression for the reaction force between the car and the road, in terms of the frictional force.
- b Resolve towards the centre of the circle.
- c Find the coefficient of friction between the car and the road.

Answer

a
$$R(\uparrow)$$
: $R\cos 15 = 1000g + F\sin 15$ Resolve vertically. There are three forces.

So
$$R = \frac{10000 + F \sin 15}{\cos 15}$$
. Determine *R*.

b
$$R\sin 15 + F\cos 15 = 1000 \times \frac{25^2}{80}$$
 Resolve horizontally towards the centre.

$$(10\,000 + F\sin 15)\tan 15 + F\cos 15 = \frac{15\,625}{2}$$

$$F(\sin 15 \tan 15 + \cos 15) = \frac{15625}{2} - 10000 \tan 15$$

This gives
$$F = 4958.105 \,\text{N}$$
, then $R = 11681.28 \,\text{N}$ Determine a numerical result for F and then for R .

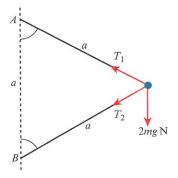
Hence,
$$\mu = 0.424$$
. Make use of $F = \mu R$.

We shall also encounter problems where there is more than one force acting towards the centre, for example, a particle with two strings attached. In Worked example 15.10, we assume that a single string with a particle attached to the centre of the string can be treated as two separate strings with different tensions.

WORKED EXAMPLE 15.10

A light, inelastic string of length 2a is tied to a fixed, vertical pole at two points, A and B. A is a distance a vertically above B. A particle of mass 2m is attached to the midpoint of the string. The particle is then set in motion to describe horizontal circles. Both parts of the string are taut at all times during the motion. Given that the speed of the particle is $\sqrt{\frac{9}{4}ga}$, find the tension in both parts of the string.

Answer



$$R(\uparrow)$$
: $T_1 \cos 60 = T_2 \cos 60 + 2mg$

$$R(\leftarrow): T_1 \sin 60 + T_2 \sin 60 = \frac{2m \times \frac{9}{4}ga}{\frac{a\sqrt{3}}{2}}$$

So
$$(T_1 + T_2) \times \frac{\sqrt{3}}{2} = \frac{9mg}{\sqrt{3}}$$
, then $T_1 + T_2 = 6mg$.

Solving $T_1 = T_2 + 4mg$ gives $T_1 - T_2 = 4mg$.

Gives $T_2 = mg$ and $T_1 = 5mg$.

By sketching a suitable diagram, we can see that we have an equilateral triangle.

The particle at the midpoint of the string splits it into two equal parts. Note that their tensions are different.

Using a sketch shows that the upper string must have a larger tension than the lower string. This information will help you when checking for errors.

Note angles are all 60°, and resolve.

Use Pythagoras or trigonometry to find the radius of the circular motion.

The force acts towards the centre.

Simplify the second equation.

Simplify the first equation and combine.

Obtain both tensions.

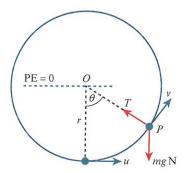
EXERCISE 15B

- 1 A particle of mass 0.9 kg is attached to a light inextensible string of length 2.4 m. The other end of the string is attached to a fixed point on the ceiling. The particle is describing horizontal circles with the string taut and making an angle of 25° with the downward vertical. Find the speed of the particle.
- 2 A particle of mass 1.6 kg is attached to a light inextensible string of length 1.4 m. The other end of the string is attached to a fixed point on the ceiling. The particle is describing horizontal circles with the string taut and making an angle of 35° with the downward vertical. Find the angular speed of the particle.
- 3 A particle of mass 2kg is attached to a light, inextensible string of length 3m. The other end of the string is attached to a fixed point on the ceiling. The particle is describing horizontal circles with the string taut and making an angle of 30° with the downward vertical. Find the angular speed of the particle.

- PS
- 4 A car is driving around a banked road inclined at 15° to the horizontal. The car has mass 1200 kg and the radius of the circular part of the road is 60 m. The coefficient of friction between the road and the tyres is 0.75, and the car is on the point of slipping up the road. Find the speed of the car.
- PS
- A particle, P, of mass 3 kg is attached to two light, inextensible strings. One string is attached at its other end to a point, A. The other string has its other end attached to a point, B. A is 4m above B. The particle makes horizontal circles such that angle PAB is 30° and angle PBA is 60°. Given that the speed of the particle is $3.6 \,\mathrm{m\,s^{-1}}$, find the tension in each string.
- A hemispherical bowl of radius a is resting in a fixed position where its rim is horizontal. A small ball of mass 2m is moving around the inside of the bowl such that the circle described by the ball is $\frac{a}{2}$ vertically below the rim of the bowl. Find the speed of the ball.
- A light, inextensible string of length 4m is threaded through a smooth ring at the point O. One end of the string has a particle, P, of mass 4kg, which is 1.2m vertically below O. Particle Q, of mass 1.6kg, is attached to the other end of the string. Particle Q is describing horizontal circles, with the string OQ making an angle of 20° with the downward vertical.
 - a If particle P does not move during the motion, find the angular speed of Q.
 - **b** Find the number of complete circles described per minute.
- A car is moving on a circular section of road where the road is banked at 25° to the horizontal. The radius of this section of road is 100 m. The car has mass 1400 kg and is travelling at $30 \,\mathrm{m\,s^{-1}}$. Given that the car is about to slip up the road, find the value of μ .
- 9 A toy plane of mass 0.4 kg is attached to one end of a light, inextensible string of length 6 m. The other end of the string is attached to the point O. The string is taut and makes an angle of 45° with the upward vertical. Find:
 - a the tension in the string
 - b the speed of the toy plane.

15.3 Vertical circles

The problems we have looked at so far involved motion in a horizontal circle but particles can also describe vertical circles.



As the object moves around the circular path, it will gain potential energy due to the gain in height. This gain in potential energy comes about when the kinetic energy reduces, so in these situations the angular speed of the particle varies. This is the main reason we use the linear speed for this type of problem.

Considering zero potential energy when the particle is at a height level with the centre of the circle, as seen in the diagram:

Initially, PE = -mgr and $KE = \frac{1}{2}mu^2$. Generally, $PE = -mgr\cos\theta$ and $KE = \frac{1}{2}mv^2$.

For the particle to complete full circles, at the top of the circle it must have KE > 0. This means that the particle continues to move after reaching the top. This idea is also reinforced by the need to have tension in the string at the very top. If there was no tension in the string, then the particle would no longer be travelling along a circular path.

So considering F = ma towards the centre at a general point, $T - mg\cos\theta = \frac{mv^2}{r}$.

Then from the principle of conservation of energy, $\frac{1}{2}mu^2 - mgr = \frac{1}{2}mv^2 - mgr\cos\theta$,

or $\frac{mv^2}{r} = \frac{mu^2}{r} - 2mg + 2mg\cos\theta$. Combining these two equations gives

 $T - mg\cos\theta = \frac{mu^2}{r} - 2mg + 2mg\cos\theta$, and if the tension is such that $T \ge 0$, then

 $\frac{mu^2}{r} \geqslant 2mg - 3mg\cos\theta.$

At the top point $\theta = 180^{\circ}$, and so $\frac{u^2}{r} \ge 5g$, giving $u \ge \sqrt{5gr}$. This is the condition for complete vertical circles for a circle of radius r, as shown in Key point 15.4.

(O) KEY POINT 15.4

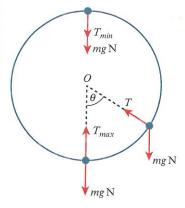
For a particle to complete vertical circles of radius r, starting from the lowest point, the speed u must satisfy the condition $u \ge \sqrt{5gr}$

WORKED EXAMPLE 15.11

A particle of mass m is attached to a light, inelastic string of length 2a. The other end of the string is attached to a fixed point, O. When the particle is resting in equilibrium below the point O, it is given a horizontal speed, u.

- a Write down the minimum value of u required to complete a full circle.
- **b** Given that $u^2 = 16ga$, find the greatest and least tension in the string.

Answer



Note that the greatest tension will always be when the particle is at the bottom of the circle.

The weight component plus the tendency of the particle to 'escape' the circular path means this is where the tension is maximum.

Similarly, at the top the tension is minimum since the speed is at a minimum and both tension and weight act downwards.

a Require
$$u_{\min} = \sqrt{5g(2a)} = \sqrt{10ga}$$
.

Use
$$u \ge \sqrt{5gr}$$
, with new radius.

$$T - mg\cos\theta = \frac{mv^2}{2a}.$$

Initially:

$$PE = -2mga, KE = \frac{1}{2}mu^2$$

Determine the energy of the system at the start and at a general point.

Use F = ma generally.

Generally: PE = $-2mga \cos \theta$, KE = $\frac{1}{2}mv^2$

By CoE:

$$\frac{1}{2}mu^2 - 2mga = \frac{1}{2}mv^2 - 2mga\cos\theta$$

At the lowest point: $\theta = 0$, $u^2 = 16ga$

$$T_{\text{max}} = 9mg \, \text{N}$$

At the highest point: $\theta = 180^{\circ}$

So
$$\left(\frac{1}{2}m\right)(16ga) - 2mga = \frac{1}{2}mv^2 + 2mga$$
,

giving $v^2 = 8ga$.

Then
$$T + mg = \frac{8mga}{2a}$$
, so $T_{\min} = 3mg \,\mathrm{N}$

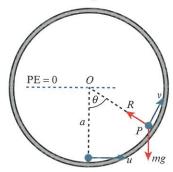
Assume conservation of energy (CoE).

Find maximum tension at the lowest point.

Use the angle at the top to find the speed.

Determine the minimum tension.

Consider a small ball placed on the inside of a smooth, hollow ring. We can model this in exactly the same way as the particle on a string.



TIP

If a particle lacks the energy to complete a full circle it can behave in two ways. If the particle has enough energy to travel beyond the horizontal midline of the circle, it will leave the surface of the ring and become a projectile. If it does not have enough energy to reach the midline, it will behave like a pendulum and eventually come to rest at the lowest point of the circle.

In the absence of a string, the ball keeping contact with the inner surface of the ring produces a reaction force towards the centre of the ring. Just as with the particle on a string, if the particle does not have enough energy it will simply leave the circular path. The particle still travels, but it is no longer in contact with the inner surface of the ring.

WORKED EXAMPLE 15.12

A particle, P, of mass m is resting on the inside of a smooth, circular ring, with centre O and radius a. The particle is projected from the lowest point with a horizontal speed of $u^2 = \frac{5}{2}ga$. Find the angle between the line OP and the downward vertical at the point where the particle leaves the surface of the ring.

Answer

Initially:
$$PE = -mga$$
, $KE = \left(\frac{1}{2}m\right)\left(\frac{5}{2}ga\right) = \frac{5}{4}mga$

Determine the energy at the start and at a general point.

Generally:
$$PE = -mga \cos \theta$$
, $KE = \frac{1}{2}mv^2$

By CoE:
$$\frac{1}{4}mga + mga\cos\theta = \frac{1}{2}mv^2 \quad (1)$$

Use conservation of energy.

Using
$$F = ma$$
, $R - mg \cos \theta = \frac{mv^2}{a}$. (2)

Resolve horizontally, towards the centre.

So
$$R - mg \cos \theta = \frac{1}{2}mg + 2mg \cos \theta$$
.

Combine the equations.

Hence,
$$R = \frac{1}{2}mg + 3mg\cos\theta$$
.

When the particle leaves its circular path:

$$R = 0 \Rightarrow \cos \theta = -\frac{1}{6}$$

Let the reaction be zero to find the point where the particle leaves the surface.

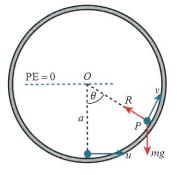
$$\theta = 99.6^{\circ}$$

Determine the angle at that point.

WORKED EXAMPLE 15.13

A particle, P, of mass m is resting on the inside of a smooth, circular ring, with centre O and radius a. The particle is projected from the lowest point with a horizontal speed of $u^2 = 3ga$. Find the greatest height above O achieved by the particle.

Answer



Label your diagram with the same notation and references as in the question.

By CoE: $mg + 2mg\cos\theta = \frac{mv^2}{a}$

Use conservation of energy and multiply through by $\frac{2}{a}$

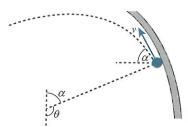
Using F = ma, $R - mg \cos \theta = \frac{mv^2}{a}$.

Resolve towards the centre.

So $R - mg \cos \theta = mg + 2mg \cos \theta$.

R = 0 gives $\cos \theta = -\frac{1}{3}$. With this angle $v = \sqrt{\frac{1}{3}ga}$.

Let the reaction force be zero to determine the angle and speed at the point when the particle leaves the surface of the ring.



As the particle leaves the inner surface, note that the angle α is important for the height of the particle at that point, as well as the component form of the speed.

Although it looks as if the particle is still moving around the circle, its path is a tangent to the circle. After this point, the particle will fall off the ring and travel as a projectile under gravity.

Since $\cos \alpha = \frac{1}{3}$, we can split the speed into components

State $\cos \alpha$ and use it to find $\sin \alpha$.

 $v_y = \sqrt{\frac{1}{3}ga} \sin \alpha = \frac{2\sqrt{2}}{3} \sqrt{\frac{1}{3}ga}$ (Only need \uparrow .)

Determine the vertical speed component.

Using $v^2 = u^2 + 2as$, so $0 = \frac{8}{27}ga - 2gs$,

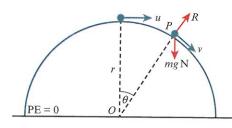
Speed at the highest point is zero.

giving $s = \frac{4}{27}a$.

So the total height above O is $a \cos \alpha + s = \frac{13}{27}a$.

State the greatest height achieved above O.

Instead of travelling on the inside or a circular surface, particles can travel on the outside. Consider a hemispherical shell resting on a horizontal surface with its plane face at the bottom. The hemisphere is smooth and has radius r. At the top of the hemisphere is a particle of mass m. The particle is given a very slight push, to produce enough speed to get the particle moving. Let us look at the situation where the particle leaves the surface of the hemisphere.



Using conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgr - mgr\cos\theta$

If we multiply through by $\frac{2}{r}$, we get $\frac{mv^2}{r} = \frac{mu^2}{r} + 2mg - 2mg \cos \theta$.

Next use F = ma towards the centre of the hemisphere to get $mg \cos \theta - R = \frac{mv^2}{r}$

or
$$-R = \frac{mu^2}{r} + 2mg - 3mg\cos\theta$$
.

When the particle leaves the surface, R = 0 and so $\cos \theta = \frac{mu^2}{r} + 2mg$.

This result is quite useful. If $u \approx 0$ then $\cos \theta \approx \frac{2}{3}$, so even with the slightest push the particle will leave the surface of the hemisphere after 48.2°.

It also shows that when $u^2 = gr$, $\cos \theta = 1$, and so $\theta = 0$, implying the particle leaves the surface immediately after being given an initial speed of \sqrt{gr} , as shown in Key point 15.5.

(A) KEY POINT 15.5

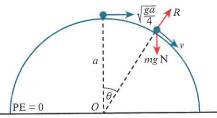
When a particle is projected from the top of a hemisphere, provided the initial speed is $0 < u < \sqrt{gr}$, the particle will turn through an angle θ , where $0 < \theta < \cos^{-1}\left(\frac{2}{3}\right)$.

WORKED EXAMPLE 15.14

A particle of mass m is projected from the top of a hemisphere of radius a. The hemisphere has its plane face in contact with the horizontal surface below it, and the hemisphere is to be considered smooth. If the initial speed at

the top is $\sqrt{\frac{1}{4}ga}$, find the speed of the particle at the moment it loses contact with the hemisphere.

Answer



It is usually easier to consider the zero potential energy line at the centre of the 'circle'. This reduces the number of combinations of distances in your calculations.

Initially: $KE = \frac{1}{8}mga$, PE = mga

Find the energy at the top.

Generally: $KE = \frac{1}{2}mv^2$, $PE = mga \cos \theta$

Determine the energy at a general point.

CoE: $\frac{mv^2}{a} = \frac{9}{4}mg - 2mg\cos\theta$

Use conservation of energy and combine this with the force towards the centre.

3/1

Using F = ma, $mg \cos \theta - R = \frac{mv^2}{a}$. Combine this with the energy

result to get $mg \cos \theta - R = \frac{9}{4}mg - 2mg \cos \theta$.

Obtain a result in R and $\cos \theta$.

When R = 0, $\cos \theta = \frac{3}{4}$ and so $v = \sqrt{\frac{3}{4}ga}$.

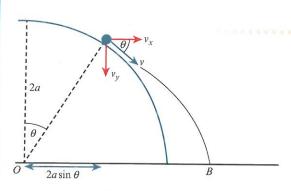
Find the speed at the point of loss of contact.

WORKED EXAMPLE 15.15

A smooth hemisphere of radius 2a is fixed with its plane face against a horizontal surface. The centre of the plane face is denoted as O. A particle of mass m is placed at the top of the hemisphere. The particle is then projected horizontally with speed $\sqrt{\frac{2}{3}} a a$. The particle first lands on the hemisphere.

horizontally with speed $\sqrt{\frac{2}{3}}ga$. The particle first lands on the horizontal surface at the point B. Determine the distance OB.

Answer



The particle will lose contact with the hemisphere at some point.

You can draw two diagrams: one for the initial calculations and one for when the particle leaves the surface.

Initially: $KE = \frac{1}{3}mga$, PE = 2mga

Generally: $KE = \frac{1}{2}mv^2$, $PE = 2mga \cos \theta$

CoE: $\frac{mv^2}{2a} = \frac{7}{3}mg - 2mg\cos\theta$

Using F = ma, $mg \cos \theta - R = \frac{mv^2}{2a}$

or $mg \cos \theta - R = \frac{7}{3}mg - 2mg \cos \theta$.

So $R = 0 \Rightarrow \cos \theta = \frac{7}{9}$. We also have $\sin \theta = \frac{4\sqrt{2}}{9}$.

Speed at leaving point is $v = \sqrt{\frac{14}{9}ga}$.

So $v_y = v \sin \theta = \frac{4\sqrt{2}}{9} \sqrt{\frac{14}{9} ga}$.

Determine the energy at the two points, as for similar problems.

Conservation of energy.

For forces towards the centre, remember that the component of the weight is greater than the reaction force.

Let the reaction force be zero to find the angle at separation.

Determine the speed at that point.

Use $s = ut + \frac{1}{2}at^2$ vertically to find the time taken to reach

Determine the time to reach the ground, where $s_v = 2a\cos\theta$.

the ground, so $\frac{14}{9}a = \sqrt{\frac{448}{729}}gat + \frac{1}{2}gt^2$.

Solving this leads to $t = 1.146 \sqrt{\frac{a}{g}}$,

so
$$v_x = v \cos \theta = \frac{7}{9} \sqrt{\frac{14}{9} ga}$$
.

Use
$$s = ut$$
 to give $s = 1.146\sqrt{\frac{a}{g}}\sqrt{\frac{686}{729}ga} = 1.112a$.

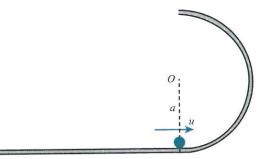
Substitute into s = ut for the horizontal distance covered.

So *OB* is
$$2a \times \frac{4\sqrt{2}}{9} + 1.112a = 2.37a$$
.

Add this distance to the initial distance covered $(2a\sin\theta)$ before losing contact.

WORKED EXAMPLE 15.16

A particle, P, of mass m, is at rest on the lowest part of a semicircular piece of metal. The semicircle has radius aand it is completely smooth. The particle is projected with a horizontal speed of $\sqrt{8ga}$.



a Find the speed of the particle when it is just about to lose contact with the semicircle.

The particle then travels as a projectile until it hits the ground.

Find the horizontal distance between O and the point where the particle hits the ground.

Answer

a Initially: PE = -mga, KE = 4mga

Generally: PE = $-mga \cos \theta$, KE = $\frac{1}{2}mv^2$

CoE: $\frac{mv^2}{a} = 6mg + 2mg\cos\theta$

From here, $\theta = 180^{\circ}$ leads to $v = 2\sqrt{ga}$.

Determine the energy at the start and at a general point, using O as the level of zero potential energy.

Use conservation of energy.

Use the angle at the top to find the speed.

373

b Distance to the ground is 2a, so $2a = 0 + \frac{1}{2}gt^2$.

Find the time to reach the ground below, noting that $v_y = 0$ at this point.

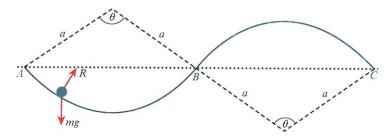
So
$$t = \sqrt{\frac{4a}{g}}$$
.

Then using
$$s = ut$$
, $s = 2\sqrt{ga} \times \sqrt{\frac{4a}{g}} = 4a$.

Use that value of the time to determine the horizontal distance travelled.

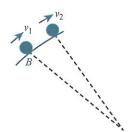
Hence, particle lands a horizontal distance of 4a from O.

We next need to look at special cases of the motion on the outer surface of an arc of a circle. Consider a track that consists of two circular arcs, as shown in the following diagram.



Imagine a particle starts at A and moves along the track to the point B, which is at the same horizontal level as A. The particle cannot lose contact with the track between A and B. The particle can, however, leave the track between B and C, providing it has enough energy to do so.

Let's look more closely at point B.



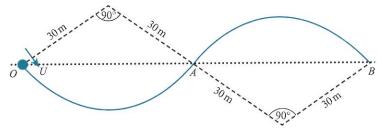
As the particle reaches B it has speed v_1 , and as it climbs the arc BC the speed will reduce to v_2 . This loss in kinetic energy will inhibit the particle's chance to escape from the surface. So if the particle does not escape the arc BC at the point B, it will never escape.

EXPLORE 15.1

For the previous example, discuss the different ways in which a particle could be projected from A and reach C.

WORKED EXAMPLE 15.17

A particle is projected from the point O with speed U along a track that is made up of two identical quarter circles, as shown in the diagram. Find the range of values of U such that the particle reaches B without losing contact with the track.



Answer

First note that $U_O = U_A$.

Since these are at the same level, the speeds are the same.

At A: KE =
$$\frac{1}{2}mU^2$$
, PE = $mg \times 30 \cos 45 = 15\sqrt{2}mg$

Determine the energy at A and at the top of the arc, using the centre of the circle that forms arc AB as the level of zero potential energy.

To just reach the top of arc AB,
$$KE = \frac{1}{2} mv^2$$
, $PE = 30mg$.

If the particle just reaches the top of the arc, it can continue down to B.

CoE:
$$\frac{1}{2}mv^2 = \frac{1}{2}mU^2 + 15\sqrt{2}mg - 30mg$$

Use conservation of energy and let the speed be positive only at the top of the arc.

Need
$$v > 0$$
, $\frac{1}{2}mU^2 > 30mg - 15\sqrt{2}mg$; hence, $U > 13.3 \,\mathrm{m \, s^{-1}}$.

This is the minimum speed required to reach B.

Using F = ma, $mg \cos \theta - R = \frac{mU^2}{30}$; let $\theta = 45^\circ$ and assume the particle will leave the track (R = 0).

Resolve towards the centre. Assume the particle leaves the track at A.

So $mg \cos 45 = \frac{mU^2}{30}$, and so $U^2 = 212.13$.

Determine the speed required to leave the track.

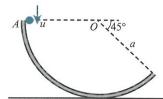
Therefore, $13.3 \,\mathrm{m \, s^{-1}} < U < 14.6 \,\mathrm{m \, s^{-1}}$.

State the range of values for U.

EXERCISE 15C

- 1 A particle of mass m is attached to the end of a light inelastic string of length 1.5a. The other end of the string is attached to a fixed point O. When the particle is resting in equilibrium, it is given a horizontal speed of \sqrt{ga} . Find the angle turned through when the particle comes to instantaneous rest.
- A particle of mass m is attached to the end of a light inelastic string of length 1.2a. The other end of the string is attached to a fixed point O. When the particle is resting in equilibrium, it is given a horizontal speed of $\sqrt{5ga}$. Find the angle turned through when the tension in the string is zero.

- A smooth hemisphere of radius 1.6a is placed plane face down and fixed onto a horizontal plane. A particle of mass 2m is placed on the top of the hemisphere and projected with speed $\sqrt{\frac{1}{4}ga}$. As the particle travels down the curved surface of the hemisphere it turns through an angle θ , where θ is measured from the vertical through the centre of the hemisphere. Find the value of θ when the particle loses contact with the hemisphere.
- A particle of mass 2m is attached to the end of a light, inelastic string of length a. The other end of the string is attached to a fixed point, O. When the particle is resting in equilibrium, it is given a horizontal speed of $\sqrt{7ga}$ and consequently describes vertical circles. Find the greatest and least tension in the string.
- M PS 5 A particle of mass m is attached to a light, inextensible string of length a. With the other end of the string attached to a fixed point, A, the particle rests in equilibrium. The particle is then given an initial horizontal speed u. Find the condition on u such that the particle never leaves the circular path, but never completes full circles.
 - M A particle of mass 3 kg is resting on the inside of a smooth, circular hoop of radius 2 m. From the bottom position, the particle is projected horizontally with speed 8 m s⁻¹. Find the greatest height achieved by the particle from the point where it is projected.
 - A smooth hemisphere, of radius 1.5a, is placed plane face down and fixed onto a horizontal plane. A particle, of mass m, is placed on the top of the hemisphere and projected with speed $\sqrt{\frac{3}{4}ga}$. As the particle travels down the curved surface of the hemisphere the particle turns through an angle θ . Find the value of θ when the particle loses contact with the hemisphere.
 - A particle of mass m is resting on the inside of a smooth, circular hoop of radius 3a. The particle is then projected from the lowest point, with horizontal speed u.
 - a State the minimum speed required to complete vertical circles.
 - b The particle is projected with speed $\sqrt{12ga}$. Find the speed when the particle has turned through 120°.
 - c Find also the reaction force exerted on the particle when the angle is 120°.
 - 9 A particle is held at rest on a smooth, circular track, which is an arc of radius a. The track is standing in a vertical plane, and it is fixed to a horizontal surface. The points O and A are such that OA is horizontal, as shown in the diagram. The particle is released from rest and proceeds to slide down the track.
 - a Find the speed of the particle at the point where it is just about to lose contact with the track.
 - b Find the greatest reaction force on the particle while it is in contact with the track.



- 10 A string of length 2a is attached to a point O and has a particle of mass 3m attached to the other end. The particle is resting in equilibrium. The particle is then projected with horizontal speed $\sqrt{15ga}$.
 - a Find the speed of the particle at the highest point of its circular path.
 - b Find the difference between the greatest and least tension in the string.

WORKED PAST PAPER QUESTION

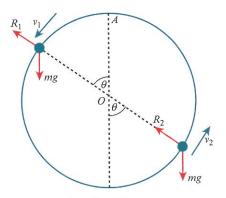
A small bead (B) of mass m is threaded on a smooth wire fixed in a vertical plane. The wire forms a circle of radius a and centre O. The highest point of the circle is A. The bead is slightly displaced from rest at A. When

angle $AOB = \theta$, where $\theta < \cos^{-1}\left(\frac{2}{3}\right)$, the force exerted on the bead by the wire has magnitude R_1 . When angle

 $AOB = \pi + \theta$, the force exerted on the bead by the wire has magnitude R_2 . Show that $R_2 - R_1 = 4mg$.

Cambridge International AS & A Level Further Mathematics 9231 Paper 2 Q2 November 2008

Answer



Consider the potential energy equal to zero at the level of O.

At the higher point: $mg \cos \theta - R_1 = \frac{mv_1^2}{a}$

$$PE = mga\cos\theta$$
, $KE = \frac{1}{2}mv_1^2$

At the lower point: $R_2 - mg \cos \theta = \frac{mv_2^2}{a}$

$$PE = -mga\cos\theta, KE = \frac{1}{2}mv_2^2$$

Adding the force equations gives $R_2 - R_1 = \frac{m}{a}(v_1^2 + v_2^2)$.

Now at A: KE = 0, PE = mga, so by conservation of energy, $mga = mga\cos\theta + \frac{1}{2}mv_1^2$, leading to $v_1^2 = 2ga(1 - \cos\theta)$.

Also by conservation of energy, $mga = -mga\cos\theta + \frac{1}{2}mv_2^2$, so $v_2^2 = 2ga(1 + \cos\theta)$.

Hence, $R_2 - R_1 = \frac{m}{a}(2ga - 2ga\cos\theta + 2ga + 2ga\cos\theta)$, which leads to $R_2 - R_1 = 4mg$.

Checklist of learning and understanding

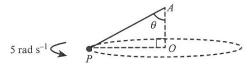
Governing equations for horizontal and vertical circles:

- For particles moving in circular paths of radius r, $v = r\omega$, where v is the linear speed and ω is the angular speed.
- For the acceleration towards the centre, $a = r\omega^2$ or $a = \frac{v^2}{r}$.
- The time for each complete circle is given by $\frac{2\pi}{\omega}$

For vertical circles:

- Using F = ma towards the centre is generally of the form $T mg\cos\theta = \frac{mv^2}{r}$ for strings and $R mg\cos\theta = \frac{mv^2}{r}$ for particles travelling on the inside of a circular path.
- Using F = ma towards the centre for motion on a hemisphere, $mg\cos\theta R = \frac{mv^2}{r}$.
- If the tension or reaction force is zero during motion, then the object has left the circular path and is now a projectile.
- Providing there are no resisting forces, the principle of conservation of energy states that $\frac{1}{2}mu^2 + PE_A = \frac{1}{2}mv^2 + PE_B, \text{ where } A \text{ and } B \text{ are usually the starting point and a general point of the motion.}$

One end of a light inextensible string is attached to a fixed point A and the other end of the string is attached to a particle P. The particle P moves with constant angular speed 5 rad s⁻¹ in a horizontal circle which has its centre O vertically below A. The string makes an angle θ with the vertical (see diagram). The tension in the string is three times the weight of P.



- i Show that the length of the string is 1.2 m.
- ii Find the speed of P.

END-OF-CHAPTER REVIEW EXERCISE 15

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q3 June 2015



- A smooth bead B of mass 0.3 kg is threaded on a light inextensible string of length 0.9 m. One end of the string is attached to a fixed point A, and the other end of the string is attached to a fixed point C which is vertically below A. The tension in the string is TN, and the bead rotates with angular speed ω rad s⁻¹ in a horizontal circle about the vertical axis through A and C.
 - i Given that B moves in a circle with centre C and radius 0.2 m, calculate ω , and hence find the kinetic energy of B.
 - ii Given instead that angle $ABC = 90^{\circ}$, and that AB makes an angle $\tan^{-1}\left(\frac{1}{2}\right)$ with the vertical, calculate T and ω .

Cambridge International AS & A Level Mathematics 9709 Paper 52 Q6 November 2011



A particle P of mass m is projected horizontally with speed $\sqrt{\frac{7}{2}ga}$ from the lowest point of the inside of a fixed hollow smooth sphere of internal radius a and centre O. The angle between OP and the downward vertical at O is denoted by θ . Show that, as long as P remains in contact with the inner surface of the sphere, the magnitude of the reaction between the sphere and the particle is $\frac{3}{2}mg(1+2\cos\theta)$.

Find the speed of P

- i when it loses contact with the sphere,
- ii when, in the subsequent motion, it passes through the horizontal plane containing O. (You may assume that this happens before P comes into contact with the sphere again.)

Cambridge International AS & A Level Mathematics 9231 Paper 21 Q3 June 2012

In this chapter, you will learn how to:

- use Hooke's law as a model to relate the force in an elastic string or spring with the extension or compression
- use the formula to calculate the elastic potential energy stored in a string or spring and to solve problems involving forces due to elastic strings or springs.

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Mechanics, Chapter 2	Find the net force acting on a body.	1 Two opposing, pulling forces act on a body. Their magnitudes are 7N and 4N. If the mass of the body is 2.5 kg, and the body starts from rest, find the distance travelled by the body in six seconds.
AS & A Level Mathematics Mechanics, Chapters 8 & 9	Determine the kinetic energy and potential energy of a system, and be able to make use of the conservation of energy.	2 A particle slides down a smooth plane inclined at 30° to the horizontal. The initial speed of the particle is 3 m s ⁻¹ . Find the speed of the particle after sliding for 14 m down the plane.

What is Hooke's law?

In the 17th century, physicist Robert Hooke discovered that, for part of the **deformation** of springs, the extension of the spring is directly proportional to the force used to deform it. We know this relationship as Hooke's law. It applies to the deformation of more than just springs. It can be used for a range of applications, from inflating balloons for a birthday party to calculating the amount of sway in a skyscraper in high winds.

In this chapter, we will look at questions in which strings are no longer inextensible. The force required to stretch a string or spring will relate directly to its **extension**, that is how much it lengthens. We shall also solve problems about more complicated systems, by considering the energy stored in a spring. We shall use the symbols explained in Key point 16.1.

(A) KEY POINT 16.1

In this chapter, we shall use the symbols T for tension, x for the length of the extension, l for the natural length of a spring, and λ for the modulus of elasticity. Note that λ is measured in N.

Unless stated otherwise, $g = 10 \,\mathrm{m \, s^{-2}}$.

16.1 Hooke's law

In previous work we have worked on problems involving connected particles or particles suspended by strings. The length of these strings did not change: they were considered inelastic, or **inextensible**.

For elastic strings, the force required to stretch the string is directly proportional to the length by which the string is extended. This gives us the relationship $T \propto x$, usually written as T = kx. The constant k varies between different materials and it also depends on the natural length of the string. We therefore use the form $T = \frac{\lambda x}{l}$. Here, λ is the modulus of elasticity, which tells us how stretchable the material is, and l is the natural length of the spring. This relationship is known as Hooke's law. We tend to use the constant $\frac{\lambda}{l}$, as shown in Key point 16.2.

381

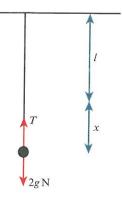
(C) KEY POINT 16.2

It is better to use the constant $\frac{\lambda}{l}$ than k as λ has the same value whatever the length of the string.

Consider a light, elastic string of length 1 m. One end of the string is attached to a ceiling, and the other end is attached to a particle of mass 2 kg. If the system is at rest in equilibrium, and the string has modulus of elasticity 50 N, find the length of the extended string.

For this example, first we resolve vertically so that T = 2g N. Then using Hooke's law, $\frac{\lambda x}{l} = 2g$. Substituting the given values gives $\frac{50 \times x}{1} = 2g$; hence, x = 0.4 m. This means the length of the extended string is 1 + 0.4 = 1.4 m.

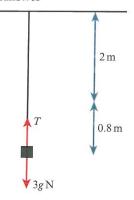
In mechanics, a good diagram is the best way to start a question. You are strongly advised to try to sketch a diagram for every question you attempt.



WORKED EXAMPLE 16.1

A block of 3 kg is attached to one end of a light, elastic string of natural length 2 m. The other end of the string is attached to a ceiling, and the string and block are at rest in equilibrium. Given that the extension in the string is $0.8 \,\mathrm{m}$, find the value of λ .

Answer



Start with a clear diagram.

Ensure all information is added.

Label distances and forces.

$$\frac{\lambda \times 0.8}{2} = 38$$

$$\lambda = 7.5g = 75 \,\mathrm{N}$$

Use Hooke's law.

Determine λ .

We shall also encounter problems involving elastic springs.

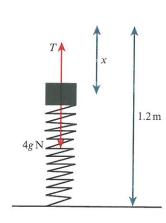
Springs can be stretched and are modelled in exactly the same way as elastic strings but with one major difference. A spring can also be **compressed**.

Consider a light spring of natural length $1.2 \,\mathrm{m}$ that is fixed to a horizontal floor so that the spring stands vertically. A mass of $4 \,\mathrm{kg}$ is placed on top of the spring so that the spring compresses by a distance x.

Given that the modulus of elasticity is $45 \,\mathrm{N}$, can we find the value of x?

Instead of tension, there is a force known as thrust, due to the spring resisting compression.

Using Hooke's law,
$$4g = \frac{45 \times x}{1.2}$$
. Hence, $x = 1.07$ m.



Let us consider stretching a spring. A light spring of unknown natural length is fixed to a ceiling at one end, with a mass of 0.5 kg attached to the other end. The spring is allowed to hang in equilibrium such that the length of the spring is now 12 cm.

If the modulus of elasticity is 10 N, find the natural length of the spring.

Since l and x are unknown, it is best to write first that l + x = 0.12. Using Hooke's law we know that $0.5g = \frac{10 \times x}{l}$, or x = 0.5l. Then 1.5l = 0.12, which gives l = 0.08 m.

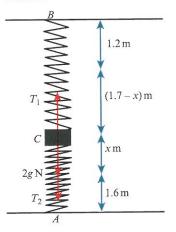
WORKED EXAMPLE 16.2

A light spring, of natural length 1.6 m and modulus of elasticity 40 N, is attached to a floor at point A. The other end of the spring is attached to a block, C, of mass 2 kg. The same block is attached to a second light spring, of natural length 1.2 m and modulus of elasticity 25 N, which is then attached to a ceiling at point B.

Both springs are vertical. A and C are vertically below B.

Given that AB is 4.5 m, find the distance AC.

Answer



Draw a diagram.

Label all forces and lengths.

Take care that you label the extension of the top spring and the compression of the bottom spring.

$$R(\uparrow)$$
: $T_1 = T_2 + 2g$

$$T_1 = \frac{25 \times (1.7 - x)}{1.2}$$
$$T_2 + 2g = \frac{40 \times x}{1.6} + 20$$

So
$$\frac{25 \times 1.7}{1.2} - 20 = \frac{40x}{1.6} + \frac{25x}{1.2}$$

Hence, $x = 0.336 \,\text{m}$ and AC is 1.94 m.

Resolve the forces vertically.

Determine the upper force.

Determine the lower force.

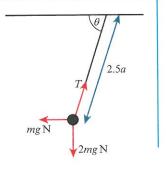
Equate the forces.

Determine x and, hence, AC.

WORKED EXAMPLE 16.3

A light, elastic string, of natural length 2a, has a mass of 2m attached to one end. The string is in a vertical plane and the system rests in equilibrium with the aid of a horizontal force of magnitude mg.

If the string is stretched to a length of 2.5a, find the tension in the string and the modulus of elasticity.



Answer

$$R(\uparrow)$$
: $T\sin\theta = 2mg$

$$R(\rightarrow)$$
: $T\cos\theta = mg$ Resolve forces horizontally.

Hence,
$$\tan \theta = \frac{2mg}{mg} = 2$$
, therefore $\sin \theta = \frac{2}{\sqrt{5}}$. Divide the equations to get $\tan \theta$. Use the triangle with sides of 1, 2, $\sqrt{5}$ to get $\sin \theta$.

Resolve forces vertically.

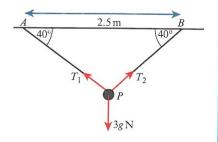
Therefore,
$$T \times \frac{2}{\sqrt{5}} = 2mg \Rightarrow T = \sqrt{5}mg$$
 N. Determine the tension.

$$T = \frac{\lambda x}{l} \Rightarrow \sqrt{5}mg = \frac{\lambda \times 0.5a}{2a}$$
 Use Hooke's law.

Therefore,
$$\lambda = 4\sqrt{5}mg$$
 N. Substitute to get λ .

WORKED EXAMPLE 16.4

A light, elastic string of natural length $1.5 \,\mathrm{m}$ is stretched between the points A and B. A particle, P, of mass $3 \,\mathrm{kg}$, is attached to the midpoint of the string. The string rests in equilibrium with the parts of the string making an angle of 40° to the horizontal, as shown in the diagram. Find the value of the modulus of elasticity.



Answer

Particle at midpoint of string
$$\Rightarrow T_1 = T_2 = T$$
. Note that the tensions are equal.

$$R(\uparrow)$$
: $2T\sin 40 = 3g$ Resolve forces vertically.

$$T = 23.336 \,\mathrm{N}$$
 Use the tension for one half of the string.

$$23.336 = \frac{\lambda x}{0.75}$$

Since
$$\cos 40 = \frac{1.25}{AP}$$
, $AP = 1.632$. Determine the new length for one half of the string.

So
$$x = 1.632 - 0.75 = 0.8818 \,\mathrm{m}$$
. Find the extension for one half of the string.

Hence,
$$\lambda = \frac{23.336 \times 0.75}{0.8818} = 19.8 \,\text{N}$$
. Determine the modulus of elasticity.

EXERCISE 16A

- 1 A light elastic string of natural length l and modulus of elasticity λ is stretched by a force T, causing the string to extend. In each of the following cases, work out the unknown value.
 - a T = 15 N, $\lambda = 40 \text{ N}$, l = 1.2 m. Find the extension.
 - **b** $T = 25 \,\mathrm{N}, \, l = 1.5 \,\mathrm{m}, \, x = 0.5 \,\mathrm{m}$. Find the modulus of elasticity.
 - c $T = (3\lambda 11)$ N, l = 2 m, x = 0.5 m. Find the tension in the string.

TIP

It is better to use the constant $\frac{\lambda}{l}$ than k as λ has the same value whatever the length of the string.

- 2 A light string of unknown natural length is fixed to a ceiling at one end, with a mass of 0.4 kg attached to the other end. The string is allowed to hang in equilibrium such that the length of the string is now 15 cm. If the modulus of elasticity is 20 N, find the natural length of the string.
- M A light string is attached to a ceiling at the point A. The natural length of the string is 1.3 m and the string has a mass of 3m attached to the free end. The string is allowed to rest in equilibrium. Given that the string is stretched by an extra 20% in length, find the value of the modulus of elasticity.
- A light, elastic string is attached to a ceiling at the point A. The natural length of the string is 1.6 m and the string has a mass of 2m attached to the other end. The string is allowed to rest in equilibrium. Given that the string is stretched by 40% of its length, find the value of the modulus of elasticity.
- Two points, A and B, lie 3m apart on a smooth horizontal surface. A light, elastic string, of natural length 0.8 m and modulus of elasticity 70 N, is attached to A. Its other end is attached to a particle, P, of mass m. A second light, elastic string, of natural length 1.3 m and modulus of elasticity 50 N, is attached to the point B. Its other end is also attached to the particle P. The particle is allowed to rest in equilibrium. Find the distance BP.
- of natural length 1.5 m is attached to A, and at the other end it has a particle of mass 6 kg attached. A second light spring of natural length 1 m is attached to the point B. Its other end is attached to the same particle. Both springs are in the same vertical plane. Given that the modulus of elasticity of the lower spring is one-third of that of the upper spring, and that the extension of one spring is six times the compression of the other spring, find the modulus of elasticity of the lower spring.
- P P A particle, P, of mass 3.5 kg is attached to one end of a light, elastic string, of natural length 1.2 m. The other end of the string is attached to a fixed point, O. The particle is allowed to rest in equilibrium with P 1.8 m below O.
 - a Show that the modulus of elasticity λ is 7g N.
 - b The particle is now pulled horizontally to the side by a force of magnitude X N so that the angle between the string and the downward vertical is 60° . Given that the particle is at the same vertical level as it was before, find the value of X.
- 8 A particle, P, of mass 4kg is resting on a rough, horizontal table. A light, elastic string, of natural length 2m and modulus of elasticity 50 N, is attached to P. It is then passed over a smooth pulley to a second particle, Q, of mass 3 kg. Q is hanging freely below the table and the system is on the point of slipping. Determine the coefficient of friction and find the extension of the string.
 - 9 A particle of mass 2 kg is being held in equilibrium on a smooth slope by a horizontal force, P, and a light, elastic spring. The spring has modulus of elasticity 10 N and is attached to the particle and also to the slope 1.5 m up the slope from the particle. If the slope is inclined at 25°, and the force P is of magnitude 5 N, find the two possible natural lengths of the spring.





- 10 A particle of mass km is placed on top of a light, vertical spring of natural length 2a. The modulus of elasticity of the spring is 3mg and the spring is standing upright so that it lies in a vertical plane.
 - a Find the compression in the spring in terms of k and a.
 - **b** Find the value of k such that the length of the spring is halved due to the weight of the particle.

16.2 Elastic potential energy

When an elastic string or spring is stretched, or when a spring is compressed, elastic potential energy (EPE) is stored in the system. This is the energy stored in strings and springs as they are stretched or compressed.

For example, if an elastic string is stretched, then the work required to stretch the string is related to the force applied and the length of the extension, x.

Since work done is equal to force multiplied by distance, we can state that $W = \int_{x}^{x_2} F dx$.

Here the force being applied is considered over a distance $x_2 - x_1$. If we use $\frac{\lambda x}{l}$ as the

force, then the work done in stretching the string is $\int_0^x \frac{\lambda x}{l} dx = \left[\frac{\lambda x^2}{2l}\right]_0^x$. So the work required,

the elastic potential energy, can be written as $EPE = \frac{\lambda x^2}{2l}$. The units are joules, as shown in Key point 16.3.

(A) KEY POINT 16.3

Work-energy principle: If a constant force acts on a body over a given distance, then the work done by the force is equal to the energy gain of the object.

For example, a light elastic string, of natural length 1.5 m and modulus of elasticity 50 N, is stretched to 1.75 m. To find the energy stored in the string use $\frac{\lambda x^2}{2l}$.

So EPE =
$$\frac{50 \times 0.25^2}{2 \times 1.5}$$
 = 1.04 J.

WORKED EXAMPLE 16.5

Find the elastic potential energy in each of the following systems.

- a A light, elastic string of natural length 2m and modulus of elasticity 100 N, is stretched by 1.2 m.
- b A light, elastic spring, of natural length 1.8 m and modulus of elasticity 50 N, is compressed by one-third of its original length.
- c A light spring, of natural length 3a and modulus of elasticity 4mg, is stretched to 5a.

Include the units for energy.

Answer

a EPE =
$$\frac{100 \times 1.2^2}{2 \times 2}$$
 Input all values into the formula.

Determine the compression distance.

Therefore, EPE = 5J.

 $c \quad EPE = \frac{4mg \times (5a - 3a)^2}{2 \times 3a}$

Make a note of the extension in the spring.

 $=\frac{8}{3}mgaJ$

State the result.

As mentioned in Section 16.1, we may encounter questions that are more complicated. Let us consider an example where a particle of mass 4 kg is tied to two light, elastic strings.

The top string has natural length 1 m and modulus of elasticity 3g N. The bottom string has natural length 1.2 m and modulus of elasticity 2g N.

To determine the elastic energy stored in the system we must find the extension of each string. Given that AB is $4 \,\mathrm{m}$, we can work out the middle part and then determine x.

 $R(\uparrow)$: $T_1 = T_2 + 4g$, then using Hooke's law on each string and equating the forces we get

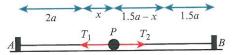
$$\frac{3g(1.8-x)}{1} = \frac{2gx}{1.2} + 4g$$
. This leads to $x = 0.3$ m.

Then add the elastic potential energy stored in each string to get

EPE =
$$\frac{3g \times 1.5^2}{2 \times 1} + \frac{2g \times 0.3^2}{2 \times 1.2} = 34.5 \text{ J}.$$

Two points, A and B, lie on a smooth horizontal surface a distance 5a apart. A light, elastic string, of natural length 2a and modulus of elasticity 3mg, is attached to A and then attached to a particle, P, of mass m. A second light, elastic string, of natural length 1.5a and modulus of elasticity 7mg, is attached to B and then attached to the same particle P. Determine the amount of EPE stored in the system.

Answer



Sketch a diagram showing the forces and appropriate lengths.

$$R(\rightarrow)$$
: $T_1 = T_2$

Resolve horizontally.

So
$$\frac{3mg \times x}{2a} = \frac{7mg(1.5a - x)}{1.5a}$$
.

Then 4.5mg x = 21mg a - 14mg x.

Simplify the equation.

Hence,
$$x = \frac{42}{37}a \approx 1.135a$$
.

Determine the extension for each string.

Then EPE =
$$\frac{37}{3mg \times (1.135a)^2} + \frac{7mg \times (0.3649a)^2}{2 \times 1.5a}$$

Sum the EPE for each string.

Work out the result.

WORKED EXAMPLE 16.7

A light, elastic string of length $2.5\,\mathrm{m}$ has modulus of elasticity $65\,\mathrm{N}$. Find the work done when stretching the string from $3\,\mathrm{m}$ to $6\,\mathrm{m}$.

Answer

For 0.5 m extension: EPE = $\frac{65 \times 0.5^2}{2 \times 2.5}$ = 3.25

Find the smaller extension EPE.

For 3.5 m extension: EPE = $\frac{65 \times 3.5^2}{2 \times 2.5}$ = 159.25

Find the larger extension EPE.

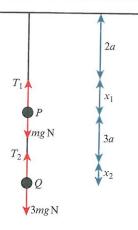
Hence, work done is 156 J.

Deduce that the energy required is the difference between the two.

WORKED EXAMPLE 16.8

A light, elastic string, of natural length 2a and modulus of elasticity 7mg, has one end tied to a point on a ceiling. The other end of the string is attached to a particle, P, of mass m. A second light, elastic string, of natural length 3a and modulus of elasticity 5mg, is tied to P. Its other end is attached to a second particle, Q, of mass 3m. Q is vertically below P.

With the system resting in equilibrium, both strings are taut. Find the elastic potential energy stored in the system.



Answer

For the upper string: $4mg = \frac{7mg \times x_1}{2a}$, so $x_1 = \frac{8}{7}a$.

Upper string holds two particles.

For lower string: $3mg = \frac{5mg \times x_2}{3a}$, so $x_2 = \frac{9}{5}a$.

Lower string holds only the lower particle.

Total EPE = $\frac{7mg \times \frac{64}{49}a^2}{2 \times 2a} + \frac{5mg \times \frac{81}{25}a^2}{2 \times 3a} = \frac{349}{70}mgaJ$

Add the two EPEs together.

i) did you know?

Most materials, including many metals and even glass, follow Hooke's law for part of their movement in tension or compression.

Any material that, when stretched or compressed, does not return to its original form is said to be plastic.

- A light elastic string of natural length l and modulus of elasticity λ is stretched by an extension x, by means of a force T, causing the string to gain elastic potential energy (EPE). Find the unknown value in each case.
 - a $\lambda = 50 \,\text{N}$, $x = 0.5 \,\text{m}$, $l = 1.5 \,\text{m}$. Find the energy stored in the string.
 - **b** EPE = 100 J, $\lambda = 60$ N, l = 1.2 m. Find the extension of the string.
 - c EPE = 50 J, x = 0.8 m, l = 1.6 m. Find the modulus of elasticity.
- Prove, using integration, that the elastic potential energy stored in a string of natural length l and modulus of elasticity λ is $\frac{\lambda x^2}{2l}$, where x is the extension of the string.
- A particle of mass 5 kg is attached to one end of a light elastic string of natural length 0.8 m. The other end of the string is attached to a fixed ceiling and the particle is allowed to rest, hanging in equilibrium. If the string is stretched by an additional 80%, determine the modulus of elasticity. Hence find the elastic potential energy stored in the string.
- A particle of mass 3 kg is attached to one end of a light, elastic string of natural length 1.2 m. The other end of the string is attached to a fixed ceiling and the particle is allowed to rest, hanging in equilibrium. If the string is stretched by an additional 50%, determine the modulus of elasticity and, hence, find the elastic potential energy stored in the string.
- The points A and B are 4.5 m apart, with A vertically above B. Particle P, of mass 2 kg, is connected to A and B by means of two light, elastic springs. The spring attached to point A has natural length 1 m and modulus of elasticity 50 N; the spring attached to B has natural length 1.4 m and modulus of elasticity 80 N. The system rests in equilibrium. Find the EPE stored in the system.
- A light elastic string has one end attached to a ceiling. The other end is attached to a particle of mass 3m. The string has natural length 2a and modulus of elasticity 6mg. The particle rests in equilibrium.
 - a Show that the extension in the string is a.
 - **b** A smaller particle of mass km is attached to the existing particle. If the EPE increase is $\frac{8}{3}mgaJ$, find the value of k.
- A light, elastic string of natural length 2.4m is stretched between two points, A and B, on a horizontal ceiling. The distance AB is 4m. The modulus of elasticity of the string is 60 N. A particle of mass 3.5 kg is attached to the midpoint of the string. The system rests in equilibrium, with both parts of the string making an angle of 30° with the ceiling.
 - a Find the extension of the string.
 - b Determine the amount of elastic potential energy stored in the string.
- A light, elastic string, of natural length $\frac{3}{2}a$ and modulus of elasticity 2mg, is attached to a ceiling at the point A. The other end is attached to a particle, P, of mass 5m. A second light, elastic string, of natural length $\frac{5}{2}a$ and modulus of elasticity 8mg, is attached to P. The other end of this string has a particle, Q, attached to it. The system is allowed to rest in equilibrium. Given that the mass of Q is 3m, find:
 - a the distance AQ
 - **b** the amount of elastic potential energy stored in the system.

388

16.3 The work-energy principle

We can also model the motion of a system that is no longer in equilibrium. For example, if a particle hanging on an elastic string is pulled down further and released, what happens next? We assume motion will happen. So in this section we look at acceleration and forces at certain points in the motion. We use kinetic energy (KE), potential energy (PE), work lost due to resistance, and elastic potential energy (EPE) to analyse the motion of objects.

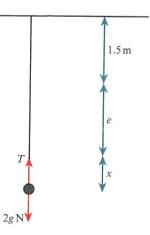
Consider a particle of mass 2 kg that is attached to a light, elastic string, of natural length 1.5 m and modulus of elasticity 4g N. The string is attached to a ceiling at its other end. The particle is allowed to rest in equilibrium before being pulled down a further 0.5 m and then released.

Suppose we want to work out the speed of the particle after it has travelled 0.4m upwards. We need to determine PE, KE and EPE at the start and finish. To do this, we must find the initial extension e of the string. We use Hooke's law to get $2g = \frac{4g \times e}{1.5}$, and so e = 0.75 m. Assign a zero potential point, then measure all potential energy relative to this point.

Initially, KE = 0, PE = 0, EPE = $\frac{4g \times (0.75 + 0.5)^2}{2 \times 1.5} = \frac{25}{12}g$ J, and then finally

KE =
$$\frac{1}{2} \times 2v^2$$
, PE = $2g \times 0.4$ and EPE = $\frac{4g \times (1.25 - 0.4)^2}{2 \times 1.5} = \frac{289}{300}g$ J. By the conservation of

energy,
$$\frac{25}{12}g = v^2 + 0.8g + \frac{289}{300}g$$
, hence $v^2 = \frac{8}{25}g$, and so $v = \frac{2\sqrt{2g}}{5} \text{ m s}^{-1}$. This is about 1.79 m s^{-1} .



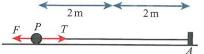
WORKED EXAMPLE 16.9

A light, elastic string, of natural length $2 \,\mathrm{m}$ and modulus of elasticity $60 \,\mathrm{N}$, has one end attached to a point, A, on a rough horizontal surface. The other end is attached to a particle P of mass $3 \,\mathrm{kg}$ that is on the table. The particle is pulled to the side so that the distance AP is $4 \,\mathrm{m}$.

- a If the coefficient of friction between the table and the particle is 0.4, find the speed as the particle passes through the point A.
- **b** Find the initial acceleration.

Answer

a



Initially: KE = 0, EPE = $\frac{60 \times 2^2}{2 \times 2}$ = 60

At A: KE = $\frac{1}{2} \times 3v^2$, EPE = 0

 $F_{\text{max}} = \frac{2}{5} \times 3g = 12$

Energy lost to friction: $W_F = 12 \times 4 = 48$

Sketch a suitable diagram.

Label the forces and add distances.

Determine the initial energy.

Don't balance the energy levels yet.

Determine the maximum frictional force.

Find the work done against friction.

Conservation of energy: $60 = \frac{3}{2}v^2 + 48$

Now balance the energy.

 $v = 2\sqrt{2} \text{ m s}^{-1}$

Determine the speed.

b F = ma, so T - F = 3a.

Use Newton's second law horizontally, followed by Hooke's law to find T.

So
$$\frac{60 \times 2}{2} - 12 = 3a$$

 $a = 16 \,\text{m s}^{-2}$

Determine the initial acceleration.

Consider a smooth, horizontal disc of radius 5a, which has its centre at the point O. A light, elastic string, of natural length 2a and modulus of elasticity 3mg, is attached to the point O. The other end of the string is attached to a particle of mass m resting on the surface of the disc. The particle remains at rest relative to the surface of the disc when the disc is rotating about the point O with angular speed ω .

If the particle is describing circles of radius 3a, how can we determine the angular speed?

First, use Hooke's law, so $T = \frac{3mg \times a}{2a} = \frac{3}{2}mg$.

Next, use Newton's second law towards the centre of the circle to get $T = mr\omega^2$, and then $\frac{3}{2}mg = m \times 3a \times \omega^2$.

So
$$\omega = \sqrt{\frac{g}{2a}} \operatorname{rad} s^{-1}$$
.

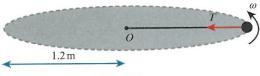
REWIND

Recall from Chapter 15 that, for particles travelling in horizontal circles, the tension component is directed towards the centre.

WORKED EXAMPLE 16.10

A smooth disc of radius 1.2 m is fixed to a horizontal plane. The centre of the disc is O. A light, elastic string, of natural length 0.8 m and modulus of elasticity 100 N, is attached to the centre O, and the other end of the string is attached to a particle of mass 0.5 kg. The particle is describing horizontal circles with constant angular speed. Given that the particle is on the point of falling off the edge of the disc, find the angular speed of the particle and the total energy in the system.

Answer



If the particle is about to fall off the edge of the disc, the radius of its circular path must be 1.2 m.

$$T = \frac{\lambda x}{l}$$
, so $T = \frac{100 \times 0.4}{0.8} = 50$.

Use Hooke's law to find the tension.

 $R(\leftarrow)$: $T = mr\omega^2$

Resolve towards the centre. Obtain the angular speed.

Then
$$50 = 0.5 \times 1.2 \times \omega^2 \Rightarrow \omega = \frac{5\sqrt{30}}{3} \,\text{rad s}^{-1}$$
.

Find the EPE in the string.

For the string: EPE =
$$\frac{100 \times 0.4^2}{2 \times 0.8} = 10$$

If a particle is

travelling along a

rough surface, then the frictional force will be

at its maximum while

the motion continues.

For the particle:

Determine the linear speed of the particle.

$$v = r\omega = 1.2 \times \frac{5\sqrt{30}}{3} = 2\sqrt{30}$$

$$KE = \frac{1}{2} \times \frac{1}{2} \times 120 = 30$$

Find the KE of the particle.

Hence, the total energy is 40 J.

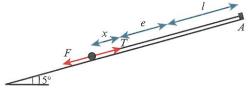
Add the energies together.

WORKED EXAMPLE 16.11

A rough slope is inclined at an angle of 15° to the horizontal. A light, elastic string is attached to the top of the slope at the point A. The string has natural length 2.4 m and modulus of elasticity 150 N. A particle of mass 1 kg is attached to the other end of the string. The particle is allowed to rest in equilibrium on the slope with the string taut.

- a Find the extension of the string.
- b The particle is pulled down a further 0.5 m and released. Given that the coefficient of friction between the particle and slope is 0.4, find the distance of the particle from A when the particle first comes to instantaneous rest.

a



Since there are two stages for the string extension, it is best to label them e and x.

Resolving up the plane: $T = g \sin 15 = \frac{150e}{2.4}$

Find the extension in the string.

So e = 0.0414 m.

b At the low point: KE = 0, PE = 0 and

Determine the energy before release.

$$EPE = \frac{150 \times 0.5414^2}{2 \times 2.4}$$
$$= 9.16$$

At the high point: KE = 0, EPE = 0 and

Find the work done against gravity.

 $PE = d \times g \sin 15$

= 2.588d

Friction: $F = 0.4 \times g \cos 15 = 3.864$

Determine the frictional force.

 $W_F = 3.864 \times d$; hence, 9.16 = 6.452d.

Find the work done against friction.

So d = 1.42.

Hence, distance = 2.4 + 0.0414 + 0.5 - 1.42= $1.52 \,\mathrm{m}$ from A Total string length minus distance travelled.

Answer

$$R(\uparrow)$$
: $T\cos 30 = 2g$; hence, $T = \frac{40}{\sqrt{3}}$. Resolve vertically to get the tension.

Using Hooke's law,
$$T = \frac{\lambda x}{l}$$
, which gives

Using Hooke's law, $T = \frac{\lambda x}{l}$, which gives: Use the tension to determine the extension in the string.

$$\frac{40}{\sqrt{3}} = \frac{50\sqrt{3} \times 3}{1.5}$$

Hence, $x = 0.4 \,\mathrm{m}$

$$R(\rightarrow): T\sin 30 = 2 \times 1.9\sin 30 \times \omega^2$$

Resolve towards the centre, noting the radius of the circle is 1.9 sin 30.

$$\frac{40}{\sqrt{3}} = 3.8 \times \omega^2$$

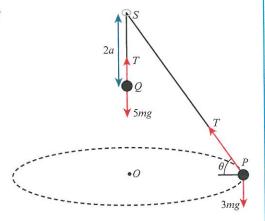
Equate results to find ω .

So
$$\omega = 2.47 \, \text{rad s}^{-1}$$

WORKED EXAMPLE 16.13

A particle, P, of mass 3m, is attached to one end of a light, elastic string of natural length 5a and modulus of elasticity 8mg. The other end of the string is threaded through a small, smooth ring, S, and a second particle, Q, of mass 5m is attached to this end. The particle P describes horizontal circles with angular speed ω . Particle Q remains stationary at a distance of 2a below the ring.

Find the extension in the string and the angular speed of Q.



For
$$Q$$
: $R(\uparrow)T = 5mg$

Particle Q is in equilibrium.

For
$$P: R(\uparrow)T\sin\theta = 3mg$$
, so $\sin\theta = \frac{3}{5}$.

The same string so the tensions are the same.

Hooke's law:
$$5mg = \frac{8mg \times e}{5a} \Rightarrow e = \frac{25}{8}a \text{ m}$$

Use Hooke's law to find the extension.

For
$$P: R(\to)T\cos\theta = 3m \times \left(3a + \frac{25}{8}a\right)\cos\theta \times \omega^2$$

Resolve towards the centre noting the radius of the circular motion is $(3a + e)\cos\theta$.

$$\cos \theta = \frac{4}{5}$$
, so this becomes $5mg \times \frac{4}{5} = 3m \times \frac{49}{8}a \times \frac{4}{5}\omega^2$

Note that $\cos \theta$ cancels.

Hence,
$$\omega = \sqrt{\frac{400}{147a}} \text{rad s}^{-1}$$
.

Evaluate the angular speed.

EXERCISE 16C

- 1 A particle of mass 0.4 kg is hanging from a light elastic string. The string has natural length 2 m and modulus of elasticity 100 N. It is held at rest with the string extended by a total of 1.2m. If the particle is released, find the kinetic energy after it has risen by 0.6 m.
- A particle of mass 2kg is attached to one end of a light elastic string of natural length 1.5 m and modulus of elasticity 50 N. The other end of the string is attached to a fixed ceiling and the particle is allowed to rest, hanging in equilibrium. The particle is then pulled down 0.3 m and released. Find the speed of the particle after it has travelled 0.4 m upwards.
- A particle, P, of mass 3 kg is attached to one end of a light, elastic spring. The other end of the spring is attached to a ceiling at the point A. The spring has natural length 1.6 m and modulus of elasticity 45 N. The system rests in equilibrium with the spring vertical.
 - a Find the extension in the spring.
 - b The particle is pulled down a further 0.6 m and released. Find the distance of the particle below the point A when it first comes to instantaneous rest.
- A light, elastic string, of natural length 2m and modulus of elasticity 50 N, is attached to two points, A and B, by its opposite ends. The points A and B are at the same horizontal level and they are 3.5m apart. A particle of mass 2 kg is attached to the midpoint of the string. The particle is then projected downwards with speed u. Given that the particle comes to instantaneous rest 2m below the level of AB, find the value of u.
- One end of a light, elastic string, of natural length 1.2m and modulus of elasticity 32 N, is attached to a fixed point, B. A particle, P, of mass 1.5 kg, is then attached to the other end of the string. The particle P is held 2m vertically above the point B and then released.
 - a Find the acceleration of P when it is 1.5 m above B.
 - **b** Find the kinetic energy of P when it is at the same horizontal level as B.
 - c Find the distance below B when the particle first comes to instantaneous rest.
- A rough, inclined plane has a string attached to it at the point C. The string has natural length 1.4m and modulus of elasticity 80 N. The string is then attached to a particle, P, of mass 4kg. P is farther down the plane than C. The particle P is then pulled down the plane so that it is 4m from C and released. Given that the coefficient of friction between the particle and the plane is 0.5, and the plane has a 30° incline, find the speed of the particle as it reaches the point C for the first time.
 - A light, elastic string, of natural length 1.8 m and modulus of elasticity 45 N, is attached to a ceiling at point G. The lower end of the string is attached to a particle, P, of mass 1.8 kg. A second light, elastic string, of natural length 0.9 m and modulus of elasticity 35 N, is attached to the particle and then to the point H, where H is 4 m vertically below G. The particle rests in equilibrium with both parts of the string taut.
 - a Find the distance of P above H.
 - **b** P is then pulled down a further 1 m and released. Find the speed when the particle is 1.5 m above H.



TIP

Always draw a fully labelled diagram. Include forces, lengths, angles and points.

- 8 A particle, P, of mass m, is moving in a horizontal circle, having centre O, with angular speed $\sqrt{\frac{g}{4a}}$. The particle is attached to one end of a light, elastic string of natural length 3a and modulus of elasticity $\frac{15}{8}mg$. The other end is attached to a fixed point, C, which is vertically above O. The string makes an angle, θ , with the downward vertical, where $\tan \theta = \frac{3}{4}$. Find the elastic potential energy in the string and the kinetic energy of the particle P.
- 9 A particle, P, of mass 3 kg, rests on a rough, horizontal table, where $\mu = \frac{2}{3}$. P is attached to a light, elastic string of natural length 2m and modulus of elasticity 49 N. The string passes over a smooth, fixed pulley at the edge of the table and is then attached to a particle, Q, of mass 1 kg. Q is held at the same level as the pulley and released. Given that P is initially 1 m from the pulley, find the speed of O at the instant P begins to move.

WORKED PAST PAPER QUESTION

1.25 m

A and B are fixed points on a smooth horizontal table. The distance AB is 2.5 m. An elastic string of natural length 0.6 m and modulus of elasticity 24 N has one end attached to the table at A, and the other end attached to a particle P of mass 0.95 kg. Another elastic string of natural length 0.9 m and modulus of elasticity 18 N has one end attached to the table at B, and the other end attached to P. The particle P is held at rest at the mid-point of AB (see diagram).

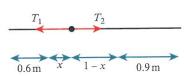
Find the tensions in the strings.

The particle is released from rest.

- Find the acceleration of P immediately after its release.
- P reaches its maximum speed at the point C. Find the distance AC.

Cambridge International AS & A Level Mathematics 9709 Paper 5 Q6 June 2007

Answer



i For the string AP: $T_1 = \frac{24 \times 0.65}{0.6} = 26 \text{ N}$ For the string *BP*: $T_2 = \frac{18 \times 0.35}{0.9} = 7 \text{ N}$

- ii Using F = ma, 26 7 = 0.95a. So $a = 20 \,\mathrm{m \, s^{-2}}$.
- iii When the maximum speed occurs, acceleration is zero, hence $T_1 = T_2$, so $\frac{24x}{0.6} = \frac{18(1-x)}{0.9}$.

Solving, $21.6x = 10.8 - 10.8x \Rightarrow x = \frac{1}{2}$; hence, $AC = 0.6 + \frac{1}{2} = \frac{14}{15}$ m.

Checklist of learning and understanding

Hooke's law:

- $T = \frac{\lambda x}{l}$, where λ is the modulus of elasticity, l is the natural length of an elastic string, and x is the extension of the string.
- Also applies to elastic springs, in which the value of x is either an extension or compression of the spring.
- When a system is resting in equilibrium, the tension is constant and proportional to the extension.

Elastic potential energy:

• Derived from the work done to extend or compress a spring or string over a certain distance.

This work is given by $\int_0^x \frac{\lambda x}{l} dx$.

- Given as $\frac{\lambda x^2}{2I}$ and measured in joules.
- Used in conjunction with kinetic and potential energy (the work-energy principle).

END-OF-CHAPTER REVIEW EXERCISE 16

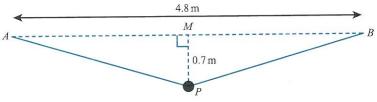


- A particle P of mass 0.28 kg is attached to the mid-point of a light elastic string of natural length 4 m. The ends of the string are attached to fixed points A and B which are at the same horizontal level and 4.8 m apart. P is released from rest at the mid-point of AB. In the subsequent motion, the acceleration of P is zero when P is at a distance 0.7 m below AB.
 - i Show that the modulus of elasticity of the string is 20 N.
 - ii Calculate the maximum speed of P.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q5 November 2010



2



A particle P of mass $0.35 \,\mathrm{kg}$ is attached to the mid-point of a light elastic string of natural length 4 m. The ends of the string are attached to fixed points A and B which are $4.8 \,\mathrm{m}$ apart at the same horizontal level. P hangs in equilibrium at a point $0.7 \,\mathrm{m}$ vertically below the mid-point M of AB (see diagram).

Find the tension in the string and hence show that the modulus of elasticity of the string is 25 N.

P is now held at rest at a point 1.8 m vertically below M, and is then released.

ii Find the speed with which P passes through M.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q6 June 2010



- The ends of a light elastic string of natural length $0.8 \,\mathrm{m}$ and modulus of elasticity $\lambda \,\mathrm{N}$ are attached to fixed points A and B which are $1.2 \,\mathrm{m}$ apart at the same horizontal level. A particle of mass $0.3 \,\mathrm{kg}$ is attached to the centre of the string, and released from rest at the mid-point of AB. The particle descends $0.32 \,\mathrm{m}$ vertically before coming to instantaneous rest.
 - i Calculate λ.
 - ii Calculate the speed of the particle when it is 0.25 m below AB.

Cambridge International AS & A Level Mathematics 9709 Paper 53 Q4 June 2011

Linear motion under a variable force

In this chapter you will learn how to:

- \blacksquare differentiate and integrate functions in terms of the time t or the displacement x
- set up and solve separable first order differential equations, using variable forces
- determine displacement, velocity or acceleration.

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Pure Mathematics 2 & 3, Chapters 4 & 5	Differentiate and integrate functions such as t^3 , $(2t - 5)^4$ and $\cos 2t$.	 a Find the derivative of 3t⁴. b Integrate sin 4t. c Differentiate e^t cos t.
AS & A Level Mathematics Pure Mathematics 2 & 3, Chapter 10	Separate the variables of a first order differential equation.	2 a Separate and then integrate $\frac{dv}{dt} = \frac{t^2}{v} \text{ to find } v = f(t).$ b Solve $e^x \frac{dv}{dx} = -2v \text{ to get } v = f(x).$

When do variable forces affect linear motion?

If an object falls through the air we like to assume that the air resistance is either negligible or constant. In fact, air resistance is a variable force that changes as the speed of an object changes. This type of variable force causes the acceleration of the object to vary, too.

Linear motion deals only with objects travelling in straight lines. When these objects experience forces that do not change, the object will either be:

- at rest or moving with a constant velocity; see Newton's first law, or
- moving with a constant acceleration, if the net force is a non-zero constant; see Newton's second law.

If the net force is non-zero, and it varies with time or distance, linear motion will be affected. This concept is the basis for motion as simple as standing up from a chair, and as complex as designing supersonic planes.

In this chapter, you will look at objects that are travelling under the influence of a variable force. This, in turn, will produce an acceleration that is variable. These systems will be:

- either set up first as differential equations such as $\frac{dv}{dt} = f(t)$ or $v \frac{dv}{dx} = g(x)$, before being solved with initial conditions or
- described in terms of displacement or velocity as functions of time.

(A) KEY POINT 17.1

In this chapter, we shall use the symbols F for force, a for acceleration, v for velocity, x for displacement and t for time.

Unless stated otherwise, $g = 10 \,\mathrm{m \, s^{-2}}$.

17.1 Acceleration with respect to time

In AS & A Level Mechanics, you saw Newton's equations of motion as well as models that involve motion with variable acceleration for basic cases.

Let us remind ourselves what we have learned previously. Acceleration is the rate of change of velocity over time. This means acceleration can be written in the form $\frac{dv}{dt}$. Another way

of confirming this result is to start with displacement, differentiate once to get $\frac{dx}{dt}$, which is velocity, then differentiate again to get $\frac{d^2x}{dt^2}$. Now, since $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$, we can see that acceleration can also be written as $\frac{dv}{dt}$.

Consider a particle that is travelling in a straight line with variable acceleration $a = -2t \,\mathrm{m}\,\mathrm{s}^{-2}$. If the initial velocity is $4 \,\mathrm{m}\,\mathrm{s}^{-1}$, can we find the velocity function in terms of time?

Start with $a = \frac{dv}{dt} = -2t$. Integrating both sides with respect to time gives $\int \frac{dv}{dt} dt = -2 \int t dt$, or $\int dv = -2 \int t dt$. Integrating, we get, $v = -t^2 + c$, and with an initial velocity of 4, this means when v = 4, t = 0 and so c = 4.

So, the velocity function in terms of time is, $v = 4 - t^2 \text{m s}^{-1}$.

WORKED EXAMPLE 17.1

A particle is travelling in a straight line with $a = \sin 2t \,\mathrm{m}\,\mathrm{s}^{-2}$. It passes through the point O with speed $v = \frac{9}{2} \,\mathrm{m}\,\mathrm{s}^{-1}$ at time $t = 0 \,\mathrm{s}$. Find:

- \mathbf{a} v in terms of t
- **b** the displacement, x, in terms of t, relative to the point O.

Answer

a Let $a = \frac{dv}{dt} = \sin 2t$, then $\int dv = \int \sin 2t \, dt$. Write down the differential equation and separate the variables.

So $v = -\frac{1}{2}\cos 2t + c$. Integrate both sides.

When $v = \frac{9}{2}$, t = 0; hence, c = 5.

Hence, $v = 5 - \frac{1}{2}\cos 2t \,\mathrm{m}\,\mathrm{s}^{-1}$.

b Let $v = \frac{dx}{dt} = 5 - \frac{1}{2}\cos 2t$. Write down the differential equation and separate the variables.

Then $\int dx = \int \left(5 - \frac{1}{2}\cos 2t\right) dt$.

Hence, $x = 5t - \frac{1}{4}\sin 2t + k$. Integrate both sides.

When x = 0, t = 0; hence, k = 0. Use the initial conditions,

So $x = 5t - \frac{1}{4}\sin 2t$ m.

Now we shall consider where this variable acceleration comes from. Consider a particle that has a force of $2t^2$ N acting in the direction of motion of the particle. If the particle has a mass of $2 \, \text{kg}$, we can work out the velocity as a function of time t.

Starting with F = ma, we have $2t^2 = 2\frac{dv}{dt}$, then $\frac{dv}{dt} = t^2$. So $\int dv = \int t^2 dt$, then $v = \frac{1}{3}t^3 + c$. If we know the initial conditions, we can determine the value of the constant c.

WORKED EXAMPLE 17.2

A particle of mass 4 kg is travelling in a straight line under the influence of a single opposing force. This force has magnitude $e^{0.5t}$ N. If the particle passes through a point, O, at time t = 0s with velocity $2 \,\mathrm{m\,s^{-1}}$, find expressions for both velocity and displacement in terms of t.

Answer

Use F = ma to get $-e^{0.5t} = 4 \frac{dv}{dt}$. Note that the force is opposing, so it is a negative force.

Then $\frac{dv}{dt} = -\frac{1}{4}e^{0.5t}$. Set up the differential equation.

So $\int dv = -\frac{1}{4} \int e^{0.5t} dt$.

Therefore, $v = -\frac{1}{2}e^{0.5t} + c$. Solve for v.

Using v = 2, t = 0 gives $c = \frac{5}{2}$.

Hence, $v = \frac{5}{2} - \frac{1}{2} e^{0.5t} \text{m s}^{-1}$. Find the first expression.

Next, let $\frac{dx}{dt} = \frac{5}{2} - \frac{1}{2} e^{0.5t}$, Set up the second equation.

then $\int dx = \frac{1}{2} \int (5 - e^{0.5t}) dt$.

So $x = \frac{5}{2}t - e^{0.5t} + k$, when measuring from O.

When x = 0, t = 0: Use the initial conditions.

And so $0 = 0 - 1 + k \Rightarrow k = 1$. Note the constant is not always 0.

Hence, $x = \frac{5}{2}t - e^{0.5t} + 1 \,\text{m}$. Find the second expression.

If you are given a velocity function such as $v = (2 - 3t)^2 - 4 \,\mathrm{m\,s^{-1}}$ and asked to find the acceleration, you can differentiate this expression using the chain rule to give $\frac{\mathrm{d}v}{\mathrm{d}t} = -3 \times 2 \times (2 - 3t)$. Hence, the acceleration is $a = -6(2 - 3t) \,\mathrm{m\,s^{-2}}$.

Similarly, if you are given $x = te^t$ and you are asked for v = f(t) or a = g(t),

then, for this example using the product rule, $v = \frac{dx}{dt} = 1 \times e^t + t \times e^t = e^t(t+1)$.

Then $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 1 \times e^t + (t+1) \times e^t = e^t(t+2).$

So $v = e^{t}(t+1) \,\mathrm{m}\,\mathrm{s}^{-1}$ and $a = e^{t}(t+2) \,\mathrm{m}\,\mathrm{s}^{-2}$.

401

WORKED EXAMPLE 17.3

A particle is travelling along a straight path such that its displacement at time t is given by the expression $x = 2\ln(t^2 + 1) - 2m$. Find the time when the velocity is at a maximum.

Answer

From
$$x = 2\ln(t^2 + 1) - 2$$
, $\frac{dx}{dt} = \frac{2 \times 2t}{t^2 + 1}$. So $v = \frac{4t}{t^2 + 1}$.

Then using the quotient rule $\frac{dv}{dt} = \frac{(t^2 + 1) \times 4 - 4t \times 2t}{(t^2 + 1)^2}$, so $a = \frac{4(1 - t^2)}{(t^2 + 1)^2}$.

 v_{max} occurs when a = 0, so $1 - t^2 = 0$ means t = 1 s.

Differentiate to get v.

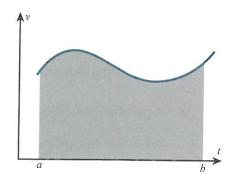
Differentiate again to get a to find when the velocity is at a maximum.

Let a = 0 to find time. Notice we discard the negative value for time.

You will have learned in AS & A Level Mechanics that the area under a velocity—time graph represents the displacement. So, to determine the area we consider the integral $\int_a^b v \, dt$, where the values of a and b are times during the motion. So $x = \int_{t_1}^{t_2} v \, dt$ gives the desired result.

For example, if we are told that a particle is such that the velocity at time t is given as $v = t^2 + 2t$, then to find the displacement of the particle after four seconds, let $x = \int_0^4 (t^2 + 2t) dt$.

So
$$x = \left[\frac{1}{3}t^3 + t^2\right]_0^4 = \frac{112}{3}$$
 m.



WORKED EXAMPLE 17.4

A particle is travelling along a straight line such that $v = t^3 - t^2 + 2 \,\mathrm{m}\,\mathrm{s}^{-1}$. Find:

- a the acceleration when t=2
- **b** the distance travelled during the third second.

Answer

- a $\frac{dv}{dt} = 3t^2 2t$, so $a = 3t^2 2t$. Differentiate to get a. When t = 2: $a = 12 - 4 = 8 \text{ m s}^{-2}$ Determine the value.
- b For the third second, we need t = 2 to t = 3. Note what 'third' second actually represents.

So
$$x = \int_{2}^{3} (t^3 - t^2 + 2) dt = \left[\frac{1}{4} t^4 - \frac{1}{3} t^3 + 2t \right]_{2}^{3}$$
. Integrate to get x .

This is $\left(\frac{1}{4} \times 81 - \frac{1}{3} \times 27 + 6 \right) - \left(\frac{1}{4} \times 16 - \frac{1}{3} \times 8 + 4 \right)$. Substitute in limits to find the value.

So $x = \frac{143}{12}$ m.

TIP

If a particle changes its direction during its motion, then the distance travelled will not be the same as the displacement.

EXPLORE 17.1

Explore functions such as $v = t^2 - t \,\mathrm{m}\,\mathrm{s}^{-1}$ for $0 \le t \le \frac{3}{2}$, and $v = t^3 - 3t^2 + 2t \,\mathrm{m}\,\mathrm{s}^{-1}$ for $0 \le t \le 2$. How would you approach this type of problem?

EXERCISE 17A

1 In each case you are given the displacement function of a particle. Find the velocity and acceleration at the time given.

a $x = 3t^2 + 5t^3$, t = 2

b $x = e^{2t} - 5t, t = 0$

c $x = 5 \ln(t+1) + t^2, t = 3$

- 2 A force of magnitude t^2 N is applied to a particle of mass 0.3 kg, resting on a smooth horizontal surface, for two seconds. Find the speed of the particle after the two seconds.
- A particle of mass $0.5 \,\mathrm{kg}$ is travelling on a smooth horizontal surface with a constant speed of $4 \,\mathrm{ms}^{-1}$. A force of magnitude kt N is applied to the particle to slow it down. Given that the particle comes to rest after three seconds, find the value of k.
- A particle is travelling with velocity function $v = \frac{1}{t^2} \text{m s}^{-1}$. Given that x = 2 m when t = 1 s, find x = f(t). What does this tell you about the displacement of the particle?

A force of magnitude (4t + 3)N is applied in the direction of motion of a particle of mass 5 kg. The particle travels in a straight line. Given that the particle is already travelling at 5 m s^{-1} when the force is initially applied, find the velocity after a further two seconds.

An opposing force of magnitude $2t^2N$ is applied to an object of mass 2 kg. At the time when the force begins to act, the object is already travelling at 3 m s^{-1} and is passing through the point O. Find expressions for the velocity of the object, and the displacement relative to the point O.

A particle is travelling in a straight line with displacement function $x = te^{-t}m$. Determine the time when the velocity is at a maximum.

- 8 A truck of mass 12000 kg is driving at a constant speed of 15 m s⁻¹. The truck driver sees a red traffic light 100 m ahead and applies the brakes. This produces a braking force of magnitude 300t² N. Will the truck stop before reaching the traffic lights?
- 9 A particle travels in a straight line with velocity $v = 2 + \sin t \, \text{m s}^{-1}$. It passes through the point O when $t = 0 \, \text{s}$. Find the displacement from O after four seconds.
- 10 A ball of mass $m \log x$ is dropped from a very high tower. Due to air resistance the ball is subject to an opposing force of magnitude mkvN, where k is a constant and v is the velocity of the ball. Show that $v = \frac{g}{k}(1 e^{-kt})$ and state the terminal speed of the ball, assuming that it does not hit the ground first.
- 11 A particle passes a point, O, with speed $12 \,\mathrm{m\,s^{-1}}$, travelling in a straight line. The acceleration of the particle is $-4t \,\mathrm{m\,s^{-2}}$. Find:
 - a the time taken for the particle to be at instantaneous rest
 - b the distance travelled during this time.



- 12 A toy rocket, of mass 1 kg, is modelled as a particle. It is launched from rest using its engines, which produce a force of size (20 - t)N and have enough fuel for five seconds. After five seconds the rocket will be subject to one force only: its weight.
 - a Show that, for $0 \le t \le 5$, $\frac{dv}{dt} = 2(5-t)$.
 - **b** Find the velocity when t = 5 s.
 - c Find the maximum height achieved, to the nearest metre.

17.2 Acceleration with respect to displacement

As well as measuring the motion of objects with time, we can also measure using displacement.

Consider a particle moving such that the acceleration is given as $a = -\frac{2}{v^2}$ m s⁻². Suppose we need to find v = f(x), given that $v = 2 \text{ m s}^{-1}$ when x = 1 m. To solve this, we must set up a differential

We cannot use $\frac{dv}{dt} = -\frac{2}{r^2}$ since the variables do not match. But if we use the chain rule on

 $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$, then $v \frac{dv}{dx} = -\frac{2}{x^2}$, as shown in Key point 17.2. Separating the variables gives

 $\int v \, dv = -\int \frac{2}{\sqrt{2}} \, dx$. Then integrating gives $\frac{1}{2}v^2 = \frac{2}{x} + c$, and using the initial conditions we

find c = 0 and $v = \frac{2}{\sqrt{100}} \text{m s}^{-1}$.

O) KEY POINT 17.2

In order to solve differential equations involving acceleration and displacement, use the acceleration form $a = v \frac{dv}{dx}$



DID YOU KNOW?

By considering an object with initial velocity u, general velocity v and constant acceleration k, you can use $\frac{dv}{dt} = k$ and $v \frac{dv}{dx} = k$ to obtain Newton's equations of motion. These are also known as SUVAT equations since they involve the variables s, u, v, a and t.

WORKED EXAMPLE 17.5

A particle is travelling in the direction Ox away from the point O with acceleration $a = 2(x - 1)^2 \,\mathrm{m}\,\mathrm{s}^{-2}$. Given that the velocity is 4 m s^{-1} when x = 1 m, find the velocity when x = 4 m.

Answer

Let
$$v \frac{dv}{dx} = 2(x-1)^2$$
, then $\int v \, dv = \int 2(x-1)^2 \, dx$.

Start with the differential equation and separate the variables.

So
$$\frac{1}{2}v^2 = \frac{2}{3}(x-1)^3 + c$$
.

Integrate both sides.

When
$$v = 4$$
, $x = 1 \Rightarrow c = 8$, so $\frac{1}{2}v^2 = \frac{2}{3}(x - 1)^3 + 8$. Use the initial conditions.

When
$$x = 4$$
, $\frac{1}{2}v^2 = \frac{2}{3} \times 27 + 8$, so $v = 2\sqrt{13} \text{ m s}^{-1}$. Determine the velocity at the given point.

WORKED EXAMPLE 17.6

A particle is subject to forces that produce an acceleration of $a = -\tan x \,\mathrm{m\,s^{-2}}$, where $0 \le x \le \frac{\pi}{3}$. The particle travels in a straight line, and when $x = 0 \,\mathrm{m}$ it passes through the point O with velocity $4 \,\mathrm{m\,s^{-1}}$.

- a Describe what happens to the particle after passing through O.
- b Find the velocity of the particle in terms of the displacement.
- c Find v when $x = \frac{\pi}{3}$ m.

Answer

- a The particle slows down as the acceleration a < 0, so the particle must decelerate becomes more negative.
- **b** $v \frac{dv}{dx} = -\tan x$, then $\int v \, dv = \int \frac{-\sin x}{\cos x} \, dx$. Separate the variables and move the negative sign inside the integral.

So
$$\frac{1}{2}v^2 = \ln|\cos x| + c$$
. Integrate both sides.

$$x = 0, v = 4, \text{ so } c = 8.$$
 Use the initial conditions.

Then
$$\frac{1}{2}v^2 = \ln|\cos x| + 8$$
.

Hence,
$$v = \sqrt{2 \ln|\cos x| + 16} \,\mathrm{m \, s^{-1}}$$
. State the expression.

c
$$v = \sqrt{2 \ln \left| \cos \frac{\pi}{3} \right|} + 16 \approx 3.82 \,\mathrm{m \, s^{-1}}.$$
 Substitute in the values.

WORKED EXAMPLE 17.7

A particle is moving along the x-axis such that its acceleration is of magnitude $2x \,\mathrm{m}\,\mathrm{s}^{-2}$ for $0 \le x \le k$. The particle passes through the point O when $x = 0 \,\mathrm{m}$, and at this point $v = 12 \,\mathrm{m}\,\mathrm{s}^{-1}$. Given that the acceleration is directed towards O, find:

- a the velocity of the particle when $x = 4 \,\mathrm{m}$
- **b** the value of k.

Answer

a Let $v \frac{dv}{dx} = -2x$. State the differential equation and separate the variables.

Hence,
$$\frac{1}{2}v^2 = -x^2 + c$$
.

- When x = 0, v = 12, so c = 72, so $\frac{1}{2}v^2 = 72 x^2$. Integrate both sides and use the values to find v = f(x).
- Leading to $v = \sqrt{144 2x^2} \,\mathrm{m}\,\mathrm{s}^{-1}$. Find the velocity at the required point.
- When x = 4, $v = 4\sqrt{7} \,\mathrm{m \, s^{-1}}$.

b Velocity function fails to work when $v^2 < 0$.

Note the condition for the function to work.

Hence, the limit of the velocity function working is when $144 = 2x^2 \Rightarrow x = \sqrt{72}$; hence, $k = \sqrt{72}$.

Determine k.

Suppose we apply a force of magnitude $\sin x$ to a particle of mass 3 kg, where x is displacement and the force is valid for $0 \le x \le \pi$.

If the particle passes through the point O with velocity 2 m s^{-1} , and the force applied is in the direction of motion, we should be able to determine the velocity when the displacement is $\frac{\pi}{2}$ m.

We start with F = ma, then $\sin x = 3v \frac{dv}{dx}$.

So
$$\int 3v \, dv = \int \sin x \, dx$$
, which means $\frac{3}{2}v^2 = -\cos x + c$.

Next, with x = 0, v = 2, we get c = 7. Then $\frac{3}{2}v^2 = 7 - \cos x$.

When
$$x = \frac{\pi}{2}$$
 we see that $\frac{3}{2}v^2 = 7$, or $v = \sqrt{\frac{14}{3}} \,\text{m s}^{-1}$.

WORKED EXAMPLE 17.8

A toy car, of mass 0.5 kg, is travelling in a straight line and passes through the point O with velocity 8 m s^{-1} . Then an opposing force of magnitude $\frac{4}{x+1} N$ acts on the car, where x is measured from the point O. Find:

- a the velocity of the particle as a function of x
- **b** the exact distance travelled when the velocity is $6 \,\mathrm{m \, s^{-1}}$.

Answer

a From
$$F = ma$$
, $-\frac{4}{x+1} = 0.5v \frac{dv}{dx}$.

Form the differential equation. Note the opposing force is negative.

Then
$$\int v \, \mathrm{d}v = -8 \int \frac{1}{x+1} \, \mathrm{d}x.$$

So
$$\frac{1}{2}v^2 = -8\ln|x+1| + c$$
.

When
$$v = 8$$
, $x = 0 \Rightarrow c = 32$

So
$$v = \sqrt{64 - 16 \ln|x + 1|} \text{ m s}^{-1}$$
.

Integrate both sides and use the given values to find v = f(x).

b Let
$$6 = \sqrt{64 - 16 \ln|x + 1|}$$
.

Substitute in v = 6 and solve for x.

Then
$$36 = 64 - 16 \ln|x + 1|$$
,

so
$$\frac{7}{4} = \ln|x+1|$$
.

Hence,
$$x = e^{\frac{7}{4}} - 1 \,\text{m}.$$

405

EXERCISE 17B

- 1 In each of the following cases, integrate the acceleration expression using $a = v \frac{dv}{dx}$ to obtain an expression for v.
 - a $a = x^2 + 3$
- **b** $a = e^{2x} x$
- c $a = 3 \frac{1}{x^2}$
- M 2 A particle of mass 2 kg is resting on a smooth horizontal surface. A force of magnitude 4x N is applied to the particle. Find the speed of the particle after it has travelled 6m.
- A particle is travelling in a straight line such that its acceleration is $3x \text{ m s}^{-2}$. Find v = f(x), given that when $v = 5 \text{ ms}^{-2}$, x = 0 m.
- 4 A force of magnitude $\frac{x}{x+3}$ N is used to drive a car of mass 1200 kg. The car starts from rest. Find the velocity of the car when x = 6 m.
- A particle moves along the x-axis with acceleration $3e^{-2x}ms^{-2}$ directed towards the origin, O. Given that P passes through O with speed $5ms^{-1}$, find an expression for the velocity in terms of x and, hence, find the terminal speed of the particle.
- 6 A particle of mass 0.25 kg is travelling in a straight line at $6 \,\mathrm{m\,s^{-1}}$ when it passes through a point, O. An opposing force of magnitude $\frac{x}{x^2+1} \,\mathrm{N}$ is then applied to the particle. Find the speed of the particle when $x=10 \,\mathrm{m}$.
- 7 A particle is travelling in a straight line with acceleration (3 + 2x) m s⁻². As it passes through a point, O, its velocity is 2 m s^{-1} . Find v^2 in terms of x.
- 8 A particle has velocity $v = kx \sin x \text{ m s}^{-1}$, where x is the distance from the point O. The particle travels in a straight line. Given that the particle has velocity $v = 2 \text{ m s}^{-1}$ when $x = \frac{3\pi}{2} \text{m}$, find the acceleration of the particle when $x = \frac{5\pi}{2} \text{m}$.
- 9 A particle is travelling along a straight line with acceleration $4\cos\frac{x}{4}\,\mathrm{m\,s^{-2}}$. As it passes through the point where $x=0\,\mathrm{m}$ its velocity is $v=7\,\mathrm{m\,s^{-1}}$.
 - a Find an expression for the velocity in terms of the displacement x m.
 - **b** Write down the exact value of the minimum velocity of the particle.
- 10 A particle of mass 2 kg experiences a force of magnitude $\left(3x \frac{1}{x^2}\right)$ N being applied in the same direction as the particle's motion. When x = 1 m, v = 4 m s⁻¹. Find v = f(x).



WORKED PAST PAPER QUESTION

A particle P of mass 0.5 kg moves on a horizontal surface along the straight line OA, in the direction from O to A. The coefficient of friction between P and the surface is 0.08. Air resistance of magnitude 0.2 ν N opposes the motion, where ν m s⁻¹ is the speed of P at time ts. The particle passes through O with speed 4 m s⁻¹ when t = 0.

- i Show that $2.5 \frac{dv}{dt} = -(v+2)$ and hence find the value of t when v = 0.
- Show that $\frac{dx}{dt} = 6e^{-0.4t} 2$, where x m is the displacement of P from O at time ts, and hence find the distance OP when v = 0.

Cambridge International AS & A Level Mathematics 9709 Paper 5 Q7 June 2008

Answer

i Start with
$$F = ma$$
, then $-F_{Fr} - 0.2v = 0.5 \frac{dv}{dt}$.

Since
$$-F_{Fr} = 0.08 \times 0.5g$$
, $-0.4 - 0.2v = 0.5 \frac{dv}{dt}$

Multiplying by 5 gives
$$2.5 \frac{dv}{dt} = -(v + 2)$$
.

Separating the variables gives
$$\int \frac{1}{v+2} dv = -\int 0.4 dt$$
, which leads to $\ln(v+2) = -0.4t + c$.

When
$$t = 0$$
, $v = 4$ and so $c = \ln 6$. So $\ln(v + 2) = -0.4t + \ln 6$.

When
$$t = 0$$
, $\ln 2 - \ln 6 = -0.4t$; hence, $t = \frac{5}{2} \ln 3 = 2.75 \text{ s}$.

ii Starting from
$$\ln(v+2) - \ln 6 = -0.4t$$
, $\frac{v+2}{6} = e^{-0.4t}$, then $v = 6e^{-0.4t} - 2$.

Since
$$v = \frac{dx}{dt}, \frac{dx}{dt} = 6e^{-0.4t} - 2$$
.

Integrating,
$$x = \int_0^{\frac{5}{2}\ln 3} (6e^{-0.4t} - 2)dt$$
, which leads to $x = [-15e^{-0.4t} - 2t]_0^{\frac{5}{2}\ln 3}$.

So the distance is 4.51 m.

Checklist of learning and understanding

Terminology:

- Velocity is the rate of change of displacement with respect to time.
- Acceleration is the rate of change of velocity with respect to time.

Notation

- $a = \frac{d^2x}{dt^2}$, the second derivative of displacement with respect to time.
- $a = \frac{dv}{dt}$, the derivative of velocity with respect to time.
- $a = v \frac{dv}{dx}$, derived from $\frac{dv}{dt}$, is the acceleration in terms of displacement.
- $v = \frac{dx}{dt}$, the first derivative of displacement.
- $x = \int v \, \mathrm{d}t$
- $v = \int a \, dt$

END-OF-CHAPTER REVIEW EXERCISE 17



- A particle P starts from rest at a point O and travels in a straight line. The acceleration of P is (15 6x) m s⁻², where x m is the displacement of P from O.
 - i Find the value of x for which P reaches its maximum velocity, and calculate this maximum velocity.
 - ii Calculate the acceleration of P when it is at instantaneous rest and x > 0.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q4 June 2011



- A cyclist and his bicycle have a total mass of $81 \,\mathrm{kg}$. The cyclist starts from rest and rides in a straight line. The cyclist exerts a constant force of $135 \,\mathrm{N}$ and the motion is opposed by a resistance of magnitude $9v \,\mathrm{N}$, where $v \,\mathrm{m} \,\mathrm{s}^{-1}$ is the cyclist's speed at time $t \,\mathrm{s}$ after starting.
 - i Show that $\frac{9}{15 v} \frac{dv}{dt} = 1$.
 - ii Solve this differential equation to show that $v = 15(1 e^{-\frac{1}{9}t})$.
 - iii Find the distance travelled by the cyclist in the first 9s of the motion.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q6 November 2010



A particle P of mass $0.25 \,\mathrm{kg}$ moves in a straight line on a smooth horizontal surface. P starts at the point O with speed $10 \,\mathrm{m\,s^{-1}}$ and moves towards a fixed point A on the line.

At time ts the displacement of P from O is x m and the velocity of P is $v \, \text{m s}^{-1}$. A resistive force of magnitude $(5 - x) \, \text{N}$ acts on P in the direction towards O.

- Form a differential equation in v and x. By solving this differential equation, show that v = 10 2x.
- ii Find x in terms of t, and hence show that the particle is always less than 5 m from O.

Cambridge International AS & A Level Mathematics 9709 Paper 51 Q7 June 2010



Chapter 18

Momentum

In this chapter you will learn how to:



- recall and use the definition of the impulse of a constant force, and relate the impulse acting on a particle to the change in its momentum
- recall Newton's experimental law, also known as the law of restitution, and the definition of the coefficient of restitution
- understand and make use of the terms elastic and inelastic
- use Newton's experimental law to solve problems involving the direct and oblique impact of a smooth sphere with a smooth surface.



PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Mechanics, Chapter 3	Write velocity in component form.	1 A particle is moving in a straight line with velocity $12 \mathrm{ms^{-1}}$ such that the angle its path makes with the x-axis is 30° anticlockwise. Find the components of the velocity in the x and y directions.
Pure Mathematics 2 & 3, Chapter 9		
AS & A Level Mathematics Mechanics, Chapter 8	Find the kinetic energy of an object that is in motion.	2 A particle of mass 2 kg is travelling with speed 4 m s^{-1} . Later in the motion it is travelling with speed 11 m s^{-1} . Find the change in kinetic energy.

What is momentum?

Any object that has mass and is in motion has momentum. If two objects are moving at the same velocity, the heavier object will have greater momentum.

In this chapter, we shall work with the quantities momentum and impulse, understand how the formulae are derived, and understand the units used for these quantities. We shall also look at collisions between bodies, especially elastic and inelastic collisions where momentum is always conserved, using the symbols defined in Key point 18.1.

$oldsymbol{\wp})$ KEY POINT 18.1

In this chapter, we shall use the symbols m for mass, u for velocity before the collision, v for velocity after the collision, P for momentum, e for coefficient of restitution, I for impulse and KE for kinetic energy. Impulse will not be examined in this course.

Newton's experimental law and the coefficient of restitution have many real-world applications. There are strict performance rules for equipment used in sports, such as golf and tennis. Manufacturers use Newton's experimental law to ensure that their sports equipment adheres to the required elastic properties and stays within the rules.

18.1 Impulse and the conservation of momentum

Consider an object of mass m, moving with velocity v. The **momentum**, P, of this object is defined as being the product of its mass and velocity. Hence, P = mv. This quantity is a vector: it can be either positive or negative.

For an object to change its momentum, a force must be applied. If that force is applied over time, then $\frac{d}{dt}(P) = \frac{d}{dt}(mv)$, then $\frac{dP}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$. If you assume that the mass does not change over time, then $\frac{dP}{dt} = m\frac{dv}{dt}$.

Changing an object's momentum requires a force. We have seen how to use the equation from Newton's second law, F = ma.

Reversing this process by integrating should give a result that relates to momentum.

Then Ft = mv - mu. This is clearly a change in momentum, which we call **impulse**. Common practice is to let I = Ft so that I = m(v - u). The units of impulse are given in Key point 18.2.

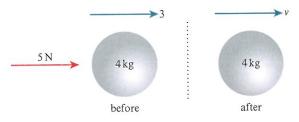
(A) KEY POINT 18.2

Impulse I is measured in newton seconds (Ns).

WORKED EXAMPLE 18.1

A particle of mass $4 \,\mathrm{kg}$ is travelling in a straight line on a smooth, horizontal surface. The particle is moving at a constant velocity of $3 \,\mathrm{m\,s^{-1}}$ when a force of magnitude $5 \,\mathrm{N}$ is applied for $0.4 \,\mathrm{s}$. Find the impulse applied to the particle and its velocity after $0.4 \,\mathrm{s}$. The particle does not change direction during its motion.

Answer



A simple clear diagram shows the before and after states of the particle.

Use I = Ft, $I = 5 \times 0.4 = 2 \text{ N s}$.

Use force and time to find the impulse.

Using I = m(v - u), 2 = 4(v - 3).

Apply impulse to the right.

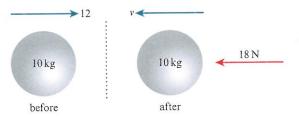
Hence, $v = 3.5 \,\mathrm{m \, s^{-1}}$.

Determine the final velocity.

WORKED EXAMPLE 18.2

A particle of mass 10 kg travels in a straight line. When the particle is travelling with velocity $12 \,\mathrm{m\,s^{-1}}$, a force of magnitude 18 N is applied to the particle for 12 s. Given that the particle's direction is reversed, find the velocity of the particle after 12 s.

Answer



Note that the particle changes direction. This implies that the momentum of the particle before the force is applied and the momentum afterwards must be in different directions.

First take positive momentum to the left.

Choose a sensible positive direction.

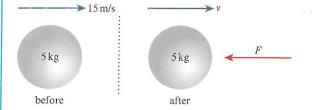
Using Ft = m(v - u), $18 \times 12 = 10(v - (-12))$. Use the impulse formula.

Then $\frac{18 \times 12}{10} - 12 = v$; hence, $v = 9.6 \,\mathrm{m\,s^{-1}}$. Determine the final velocity.

WORKED EXAMPLE 18.3

A particle of mass $5 \,\mathrm{kg}$ is travelling along a smooth, horizontal surface with velocity $15 \,\mathrm{m\,s^{-1}}$. Given that the force, F, applied over $4 \,\mathrm{s}$ reduces the velocity by $8 \,\mathrm{m\,s^{-1}}$, and that the particle still travels in the same direction, find the force F.

Answer



Note that the velocity is reduced, so F is in the opposite direction to v.

Use Ft = m(v - u) to the right.

Take positive momentum to the right.

$$-4F = 5(v - 15)$$

State the value of v.

Since v = 7, then 4F = 40.

Hence, $F = 10 \,\mathrm{N}$. Determine F.

When two particles collide, whether they remain separate after impact or combine to make a single object, the law of conservation of momentum states that $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$, as shown in Key point 18.3. m_1 , m_2 are the masses of the particles; u_1 , u_2 are the velocities before the collision, and v_1 , v_2 are the velocities after the collision.

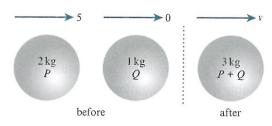
$oldsymbol{\wp}$ KEY POINT 18.3

The law of conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Consider a particle, P, of mass 2 kg travelling on a smooth, horizontal surface with velocity $5 \,\mathrm{m\,s^{-1}}$. This particle collides with a second particle, Q, of mass 1 kg, which is at rest. The particles stick together when they collide. The particles are said to coalesce.

If we draw a diagram to explain what happens, then we can attempt to find the velocity of the particle afterwards.



So using conservation of momentum in the direction of motion, $2 \times 5 + 0 = (2 + 1)v$.

Hence the velocity of the combined particles afterwards is $v = \frac{10}{3} \text{m s}^{-1}$.

This method comes from the equation $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$, as shown in Key point 18.3. Here, each particle has its own momentum and, since momentum is conserved, both sides must balance.

In this simple case, the particles coalesce and the right side becomes:

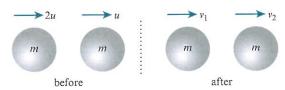
$$m_1 v + m_2 v = (m_1 + m_2) v$$

O KEY POINT 18.4

When two particles collide, the amount of kinetic energy (KE) lost can be found by comparing the KE before and after the collision. If the collision is perfectly elastic, we can use conservation of energy to equate KE before and after the collision.

When two particles collide and coalesce, or stick together, they are said to be inelastic. That means there is no bounce between them. If two particles collide and there is no loss in kinetic energy, the collision is said to be **perfectly elastic**. We can use conservation of energy, as shown in Key point 18.4.

Consider the example of two particles that are both of mass m. They are travelling with velocities 2u and u, as shown. To find a relationship for the velocities after a perfectly elastic collision, we must first find the kinetic energy before they collide.



$$KE_{before} = \frac{1}{2}m(2u)^2 + \frac{1}{2}mu^2 = \frac{5}{2}mu^2$$
, and $KE_{after} = \frac{1}{2}m(v_1^2 + v_2^2)$.

Using conservation of energy means $5u^2 = v_1^2 + v_2^2$ is the equation we need to solve. There are many solutions to this, so we shall not be able to solve this at present.

To help us deal with this, we are going to use **Newton's experimental law**, which is also known as Newton's law of restitution.

This law states that the constant $e = \frac{\text{speed of separation}}{\text{speed of approach}}$, where $0 \le e \le 1$. e is called the

coefficient of restitution. When e = 0 the objects colliding have no elasticity. This means that they will coalesce.

When e = 1, the objects are said to be perfectly elastic, as in this example.

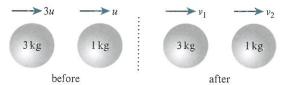
To determine v_1 and v_2 , let $e = \frac{v_2 - v_1}{2u - u}$. In this expression, $v_2 - v_1$ shows how quickly the second particle escapes the first one and 2u - u shows how quickly the particles approach each other.

So $1 = \frac{v_2 - v_1}{u}$, which means $v_2 - v_1 = u$. Then from the law of conservation of momentum we have $2mu + mu = mv_1 + mv_2$, or $3u = v_1 + v_2$. Combining these two equations gives $v_1 = u$ and $v_2 = 2u$.

WORKED EXAMPLE 18.4

Two particles, P and Q, are travelling on a smooth, horizontal plane. P has mass $3 \, \text{kg}$ and Q has mass $1 \, \text{kg}$. P has velocity 3u and Q velocity u. P then collides with Q, such that the collision is perfectly elastic. Find the velocity of each particle after the collision. Confirm that the kinetic energy is the same before and after the collision.

Answer



Using conservation of momentum: $9u + u = 3v_1 + v_2$ -----(1) Use conservation of momentum.

Using Newton's experimental law: $1 = \frac{v_2 - v_1}{3u - u}$, so $v_2 - v_1 = 2u$. (2) Use Newton's experimental law (NEL) with e = 1 as we have a perfectly elastic collision.

Combining (1) and (2) gives $v_1 = 2u \,\mathrm{m}\,\mathrm{s}^{-1}$, $v_2 = 4u \,\mathrm{m}\,\mathrm{s}^{-1}$ Solve the simultaneous equations.

Before: $KE = \frac{1}{2} \times 3 \times 9u^2 + \frac{1}{2} \times 1 \times u^2 = 14u^2J$ Find the KE before the collision.

After: $KE = \frac{1}{2} \times 3 \times 4u^2 + \frac{1}{2} \times 1 \times 16u^2 = 14u^2 J$ Confirm that the KE after the collision is the same.

When collisions are neither inelastic nor perfectly elastic, we have a coefficient of restitution that reduces the energy of the system. In most cases 0 < e < 1.

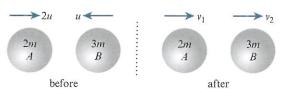
EXPLORE 18.1

Table tennis balls are made to have the value $e \approx 0.95$. Carry out some online research into different types of sporting equipment, such as footballs and tennis balls, and find out their coefficients of restitution.

WORKED EXAMPLE 18.5

Two smooth spheres of equal radius are resting on a smooth surface. Sphere A has mass 2m and sphere B has mass 3m. The coefficient of restitution between the spheres is $\frac{2}{3}$. Sphere A is projected towards B with velocity 2u and sphere B is projected towards A with velocity u. Find the velocity of each sphere after the collision. Find also the loss in kinetic energy.

Answer



It is best to make no assumptions about the directions of particles after a collision. The solutions will tell you what is happening. Newton's experimental law: $\frac{2}{3} = \frac{v_2 - v_1}{2u + u}$, so $3v_2 - 3v_1 = 6u$.

Since $v_1 < 0$, A changes direction. B also changes direction.

Before: KE = $\frac{1}{2} \times 2m \times 4u^2 + \frac{1}{2} \times 3m \times u^2 = \frac{11}{2}mu^2$

Combining gives $v_1 = -u \,\mathrm{m} \,\mathrm{s}^{-1}$, $v_2 = u \,\mathrm{m} \,\mathrm{s}^{-1}$.

Find the energy before and after the collision to determine the loss in kinetic

After: KE = $\frac{1}{2} \times 2m \times u^2 + \frac{1}{2} \times 3m \times u^2 = \frac{5}{2}mu^2$

Hence, the energy loss is $3mu^2$ J.

WORKED EXAMPLE 18.6

Two smooth spheres of equal radius are resting on a smooth surface. Sphere P has mass m and sphere Q has mass 2m. The coefficient of restitution between the spheres is e. Sphere P is projected towards Q with velocity u. Find a condition on e such that P's direction is changed.

Answer



before

Conservation of momentum: $mu = mv_1 + 2mv_2$, so $v_1 + 2v_2 = u$. Write down the two standard equations.

Newton's experimental law: $e = \frac{v_2 - v_1}{u}$, so $v_2 - v_1 = eu$.

Combining gives $v_2 = \frac{u}{3}(1+e)$, and so $v_1 = \frac{u}{3}(1-2e)$. Equate to get v_1 and v_2 in terms of e.

Consider $v_1 < 0$, and so $1 - 2e < 0 \Rightarrow e > \frac{1}{2}$.

Note P's change in direction. $v_1 < 0$ leads to the result.

DID YOU KNOW?

Newton's cradle was first demonstrated by a French physicist known as Abbe Mariotte in the 17th century. An English actor saw a toy version in 1967 and gave it the name Newton's cradle.

EXERCISE 18A

- Two particles, P and Q, are resting on a smooth, horizontal surface. P is projected towards Q with velocity 2u and they collide directly. Given that the masses of P and Q are 2m and m, respectively, and that the collision is perfectly elastic, find the velocity of P and Q afterwards.
- Two particles, P and Q, are resting on a smooth surface. They are connected by a light string of length 4 m.

 The particles are of mass 2m and 3m, respectively, and are resting 2 m apart. Q is projected directly away from P with velocity u. Given that the string is inextensible, find the velocity of the particles when the string becomes taut.
- M 3 Two identical, 1 kg, smooth spheres, A and B, of equal radius, are travelling on a smooth surface. The velocity of A is 3 m s^{-1} and the velocity of B is 2 m s^{-1} . Given that the spheres are moving in opposite directions, and that the coefficient of restitution is 0.7, find the loss in kinetic energy due to the collision.
- A particle, P, of mass 4m, is projected with velocity 2u towards another particle, Q, of mass 5m, which is at rest. Both particles are on a smooth, horizontal surface. The coefficient of restitution between the particles is e. Find the condition on e so that both particles are travelling in the same direction after the collision.
- Two particles, P and Q, are resting on a smooth, horizontal surface. Their masses are $2 \log$ and $3 \log$, respectively. Particle P is projected towards Q with velocity 5 m s^{-1} . It subsequently strikes Q and Q moves off with velocity 2.5 m s^{-1} . Find the value of e, and also find the loss in kinetic energy due to the collision.
- Two particles, P and Q, are resting on a smooth, horizontal surface. P has mass 2m and Q has mass m. P is projected towards Q with velocity u and Q is projected towards P with velocity u. Given that u has velocity u and has its direction changed due to the collision, find the value of u.
 - 7 Two smooth spheres of equal size are resting on a smooth, horizontal surface. Particle P, of mass m, is projected towards particle Q, of mass 2m, with velocity 3u. Particle Q is initially at rest. The coefficient of restitution between the spheres is e.
 - a Show that particle P has its direction changed provided $e > \frac{1}{2}$.
 - b Given that the kinetic energy after the collision is half the initial kinetic energy, find the value of e and state the velocity of P.

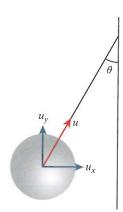
18.2 Oblique collisions and other examples

Consider a particle travelling towards a wall with velocity u at an angle of θ . If we split the velocity into component form we have u_y parallel to the wall, and u_x perpendicular to the wall.

When the particle bounces off the wall, the component u_y is unaffected by the collision, but the component u_x will be subject to the conservation of momentum.

After bouncing off the wall, the component u_x will become eu_x . Note that the angles α and θ are not the same, unless e=1.

When an object collides with a wall at an angle other than 90°, this is known as an oblique collision.



Cambridge International AS & A Level Further Mathematics: Further Mechanics

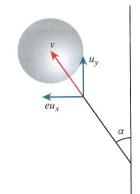
As an example, consider a particle travelling with velocity 2u, directed at an angle of 30° to a wall, with coefficient of restitution $\frac{1}{2}$ between the particle and the wall. Find the velocity of the particle after the collision.

We have $u_y = 2u \cos 30 = u\sqrt{3}$, and $u_x = 2u \sin 30 = u$.

The particle bounces off the wall so $eu_x = \frac{u}{2}$. Then for the velocity afterwards, we have

$$\sqrt{(u\sqrt{3})^2 + \left(\frac{u}{2}\right)^2}$$
, which is $\frac{u\sqrt{13}}{2}$ m s⁻¹. We can also find the angle that the particle makes

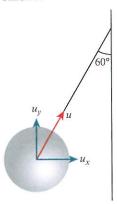
with the wall afterwards, using $\tan \alpha = \frac{eu_x}{u_y} = \frac{\frac{u}{2}}{u\sqrt{3}}$, which gives $\alpha = 16.1^\circ$.



WORKED EXAMPLE 18.7

A particle is projected along a smooth, horizontal surface with velocity u. The particle collides with a smooth, vertical wall at an angle of 60° . Given that the coefficient of restitution between the particle and the wall is 0.3, find the angle between the wall and the path of the particle after the collision.

Answer



First:
$$u_y = u \cos 60 = \frac{u}{2}$$

Split the velocity into its components.

Then
$$u_x = u \sin 60 = \frac{u\sqrt{3}}{2}$$
.

Then after bouncing,
$$eu_x = \frac{3}{10} \times \frac{u\sqrt{3}}{2} = \frac{3\sqrt{3}u}{20}$$
.

Find the component that is perpendicular to the wall after colliding.

Then
$$\tan \alpha = \frac{3\sqrt{3}}{20} \times \frac{2}{1}$$
.

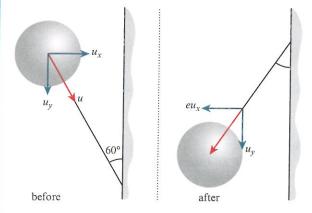
Use components to find the tangent of the angle.

So
$$\alpha = 27.5^{\circ}$$
.

Determine α .

A particle of mass 3 kg is travelling on a smooth, horizontal surface with velocity $4 \,\mathrm{m\,s^{-1}}$. The particle collides obliquely with a smooth, vertical wall where the coefficient of restitution is $\frac{1}{3}$. Given that the angle between the path of the particle and the wall before the collision is 40° , find the loss in kinetic energy due to the collision.

Answer



Parallel to wall: $u_y = 4\cos 40 = 3.0642$

Find both components before the collision.

Perpendicular to wall: $u_x = 4\sin 40 = 2.5712$

After collision, $eu_x = 0.8571$.

Find the perpendicular component after the collision.

Before: KE = $\frac{1}{2} \times 3 \times 4^2 = 24$

Determine the KE before and after the collision.

After: KE = $\frac{1}{2} \times 3 \times (3.0642^2 + 0.8571^2) = 15.186$

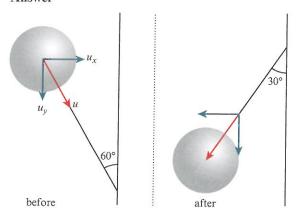
Hence, the energy loss is 8.81 J.

Determine the difference in kinetic energy.

WORKED EXAMPLE 18.9

When a smooth sphere travelling on smooth horizontal ground collides obliquely with a smooth vertical wall, it rebounds off the wall at right angles to its original direction. If the sphere is travelling with velocity u and its path makes an angle of 60° with the wall before the collision, find e, the coefficient of restitution.

Answer



Draw a clear diagram showing the situation before and after the collision.

419

Parallel: $u_y = u \cos 60 = \frac{u}{2}$ Find the velocity components before the collision.

This is unchanged after the collision.

Perpendicular: $u_x = u \sin 60 = \frac{\sqrt{3}}{2}u$ State the perpendicular component.

After the collision, $eu_x = \frac{\sqrt{3}}{2}eu$.

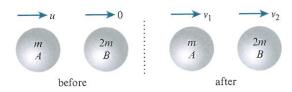
Since the angle after is 30°, $\tan 30 = \frac{\sqrt{3}}{2} eu \times \frac{2}{u}$. Use the tangent of the angle to relate the components.

So $\frac{1}{\sqrt{2}} = \sqrt{3}e \Rightarrow e = \frac{1}{3}$. Determine e

Particles can also rebound off walls at 90°.

Consider two particles, A and B, of masses m and 2m. They are resting on a smooth horizontal surface. Particle A is projected towards B with velocity u. A strikes B and then B will go on to strike a smooth vertical wall, rebounding off at 90°. If the coefficient of restitution between the particles is $\frac{7}{8}$, and the coefficient of restitution between B and the wall is $\frac{1}{3}$, can we show that there are no further collisions?

To start with, we need to draw two diagrams: one for the A-B collision, and one for B and the wall.

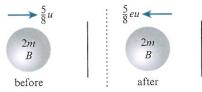


First we use conservation of momentum: $mu = mv_1 + 2mv_2$ to get $u = v_1 + 2v_2$.

Then use Newton's experimental law: $\frac{7}{8} = \frac{v_2 - v_1}{u}$, or $v_2 - v_1 = \frac{7}{8}u$

Adding these two gives $v_2 = \frac{5}{8}u$ and $v_1 = -\frac{1}{4}u$. So we know that A travels in the opposite direction away from the wall.

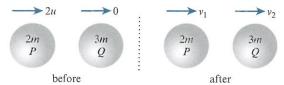
B hits the wall and bounces off with velocity $\frac{1}{3} \times \frac{5}{8}u = \frac{5}{24}u$. It is now travelling in the same direction as A.



Since $\frac{1}{4}u > \frac{5}{24}u$, B will never catch up with A, and so there will be no more collisions.

Two smooth spheres, P and Q, of masses 2m and 3m, respectively, are resting on a smooth, horizontal surface. Sphere P is projected towards Q with velocity 2u. P strikes Q and Q begins to travel towards a smooth, vertical wall. The coefficient of restitution between the spheres is e. Q strikes the wall at right angles and rebounds off the wall, where the coefficient of restitution between Q and the wall is $\frac{1}{2}$. Given that P's direction is reversed, and that Q strikes P again, find the range of values of e.

Answer



Conservation of momentum: $4mu = 2mv_1 + 3mv_2$, so $2v_1 + 3v_2 = 4u$.

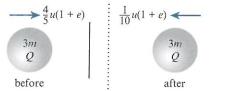
Newton's experimental law: $e = \frac{v_2 - v_1}{2u}$, so $2v_2 - 2v_1 = 4eu$.

Adding gives $5v_2 = 4u + 4eu$, or $v_2 = \frac{4}{5}u(1+e)$.

Write down the two standard equations.

Equate them to find both velocities in terms of e.

Hence, $v_1 = \frac{2}{5}u(2 - 3e)$. v_1 result to be used later.



Since $e_{\text{wall}} = \frac{1}{8}$, Q bounces off with velocity $\frac{1}{10}u(1+e)$. Allow Q to bounce off the wall.

For P to change direction 2-3e<0, so $e>\frac{2}{3}$.

For *Q* to strike *P* again, $\frac{1}{10}u(1+e) > \frac{2}{5}u(3e-2)$.

This gives $e < \frac{9}{11}$.

Hence, $\frac{2}{3} < e < \frac{9}{11}$.

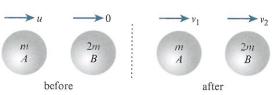
State how P changing direction affects the value of e.

Assume P is moving to the left, and allow Q to travel faster than P.

State the final range of values.

Let us now look at multiple objects colliding. We shall consider three smooth spheres, A, B and C, all of equal size, resting on a smooth, horizontal surface.

Let A have mass m, B have mass 2m and C have mass m. The coefficient of restitution between A and B is $\frac{1}{3}$, and the coefficient of restitution between B and C is $\frac{2}{3}$. So, assuming A, B and C are collinear (arranged in a straight line), let us project A towards B with velocity u.

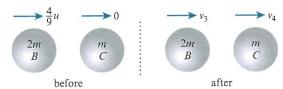


Conservation of momentum: $mu = mv_1 + 2mv_2$, so $u = v_1 + 2v_2$.

Newton's experimental law: $\frac{1}{3} = \frac{v_2 - v_1}{u}$, which leads to $v_2 - v_1 = \frac{1}{3}u$.

Equating gives $v_1 = \frac{1}{9}u$ and $v_2 = \frac{4}{9}u$.

Next, we observe B and C colliding.



So conservation of momentum: $\frac{8}{9}mu = 2mv_3 + mv_4$, so $2v_3 + v_4 = \frac{8}{9}u$.

Using Newton's experimental law: $\frac{2}{3} = \frac{v_4 - v_3}{\frac{4}{9}u}$, leading to $v_4 - v_3 = \frac{8}{27}u$.

Equating gives $v_3 = \frac{16}{81}u$ and $v_4 = \frac{40}{81}u$.

Also, according to the velocity of B, A will not collide with B again. There are no further collisions.

We can then work out the loss in kinetic energy at this point. So, initially $KE = \frac{1}{2}mu^2$, then finally we have $KE = \frac{1}{2}m\left(\frac{1}{9}u\right)^2 + \frac{1}{2} \times 2m\left(\frac{16}{81}u\right)^2 + \frac{1}{2}m\left(\frac{40}{81}u\right)^2 = \frac{731}{4374}mu^2$. So the loss in kinetic energy is $\frac{728}{2187}mu^2J$.



You are unlikely to encounter an example with more than three particles colliding.

WORKED EXAMPLE 18.11

Three particles, P, Q and R, are of masses $2 \log_2 3 \log_2 3 \log_2 4 \log$

Answer

P-Q conservation of momentum: $2 \times 5 - 3 \times 1 = 2v_1 + 3v_2$, so $2v_1 + 3v_2 = 7$.

Newton's experimental law: $\frac{3}{5} = \frac{v_2 - v_1}{6}$, and thus $v_2 - v_1 = 3.6$.

Consider the collision of *P* and *Q* first. Make sure you draw a diagram as the problems will be easier to solve.

Combining these two equations gives $v_2 = 2.84$ and $v_1 = -0.76$ (to the left).

Solve to find the velocities after the first collision.

 $\begin{array}{ccc}
& & & & & & \\
& & & & & \\
Q & & & & \\
3 \text{ kg} & & & 1 \text{ kg}
\end{array}$

Q–*R* conservation of momentum: $3 \times 2.84 = 3\nu_3 + \nu_4$, so $3\nu_3 + \nu_4 = 8.52$.

Consider the Q and R collision. Create the necessary equations.

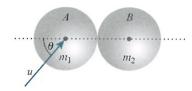
Newton's experimental law: $\frac{3}{5} = \frac{v_4 - v_3}{2.84}$ giving $v_4 - v_3 = 1.704$.

Combine to solve.

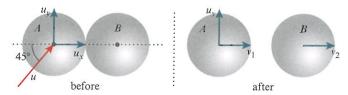
Then $v_3 = 1.704 \,\mathrm{m \, s^{-1}}$ and $v_4 = 3.408 \,\mathrm{m \, s^{-1}}$.

Calculate the velocity of R.

Finally, we look at oblique collisions between two spheres or particles. It is important to note the line of centres between the two objects. When the two particles meet, we consider the conservation of momentum in two directions: along the line of centres and perpendicular to the line of centres. The general case is shown in the diagram.



Consider two spheres, A and B, of equal radii, on a smooth horizontal surface, and with masses 2m and m, respectively. Initially, sphere B is at rest and sphere A is projected with speed u towards B. When the spheres collide, the direction of A is at 45° to the line through the centres of A and B. Given that the coefficient of restitution between the spheres is $\frac{1}{2}$, find the speeds of the particles after colliding.



Begin by considering the components of the speed of A. We notice that the only component affecting the collision is u_x . Note that $u_x = u\cos 45$, $u_y = u\sin 45$.

Using conservation of momentum, we find the vertical components will cancel to give: $2mu\cos 45 = 2mv_1 + mv_2$, which we write as $u\sqrt{2} = 2v_1 + v_2$.

Using Newton's experimental law: $\frac{1}{2} = \frac{v_2 - v_1}{u \cos 45}$ which can be written as $\frac{u\sqrt{2}}{4} = v_2 - v_1$. Combining these leads to $v_1 = \frac{u\sqrt{2}}{4}$ and $v_2 = \frac{u\sqrt{2}}{2}$. So the speed of sphere *B* after the

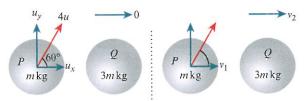
collision is $\frac{u\sqrt{2}}{2}$ m s⁻¹.

For sphere A the final speed is given by $\sqrt{u_y^2 + v_1^2} = \sqrt{\left(\frac{u\sqrt{2}}{2}\right)^2 + \left(\frac{u\sqrt{2}}{4}\right)^2}$. Hence, the speed is $\frac{u\sqrt{5}}{2\sqrt{2}}$ m s⁻¹.

We can also determine the angle of deflection caused by the collision. This is the angle that shows the change in direction travelled before and after the collision. The direction relative to the line through the centres is 45°, then after the collision the angle is $\tan^{-1}\left(\frac{u_y}{v_1}\right)$. This gives an angle of $\tan^{-1} 2 = 63.4^{\circ}$. So the angle of deflection is $63.4^{\circ} - 45^{\circ} = 18.4^{\circ}$

WORKED EXAMPLE 18.12

Two smooth spheres of equal radii, P and Q, are at rest on a smooth horizontal surface. The coefficient of restitution between them is $\frac{1}{4}$. Sphere P has mass m and sphere Q has mass 3m. Sphere P is given an initial speed of 4u and is projected in a direction such that it will collide with Q. Given that the angle between the direction of Pand the line through the centres of P and Q at the point of impact is 60° , find the angle of deflection for P.



Answer

For P: $u_x = 4u\cos 60 = 2u$, $u_y = 4u\sin 60 = 2\sqrt{3}u$

Then, along the line *PQ*: conservation of momentum: $4u\cos 60 = mv_1 + 3mv_2$, or $2u = v_1 + 3v_2$.

Newton's experimental law: $\frac{1}{4} = \frac{v_2 - v_1}{2u}$, or $v_2 - v_1 = \frac{u}{2}$.

Adding these two gives $\frac{5u}{2} = 4v_2$, and so $v_2 = \frac{5}{8}u$. Hence, $v_1 = \frac{1}{9}u$.

Before the collision, the angle between P's direction and the line through the centres is 60°.

After, it is $\tan^{-1}\left(\frac{u_y}{v_1}\right) = \tan^{-1}(16\sqrt{3})$, which is 87.9°.

Hence, the angle of deflection is 27.9°.

Work out the two components for P.

Along the line of centres we consider conservation of momentum and Newton's experimental law.

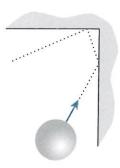
Combine the equations to find the speeds of both spheres.

Note the angle before the collision.

Determine the angle of P relative to the line of centres after the collision.

State the angle of deflection.

- 1 A small smooth ball of mass m is travelling on a smooth horizontal floor with speed u. The ball then goes on to strike a fixed smooth vertical wall at an angle $\theta = 30^{\circ}$ to the wall. The coefficient of restitution between the ball and the wall is 0.5. Find the speed of the ball after bouncing off the wall.
- 2 A small smooth ball of mass 3m is travelling on a smooth horizontal floor with speed 2u. The ball then goes on to strike a fixed smooth vertical wall at an angle $\theta = 60^{\circ}$ to the wall. The coefficient of restitution between the ball and the wall is 0.25. Find the loss in kinetic energy due to the collision with the wall.
- Three rail carriages are on a smooth, horizontal rail track. They are equally spaced apart. The carriages are labelled A, B and C, and their masses are 2m, 6m and 3m, respectively. Carriage A is projected towards carriage B with speed u, where B is in between A and C. Given that the coefficient of restitution between each carriage is 0, find the velocity of the carriages after B collides with C.
- A particle is travelling on a smooth, horizontal surface with velocity 3u. The particle collides with a smooth, vertical wall at an oblique angle of 30° to the wall. Find the angle between the path of the particle and the wall after the collision, given that $e = \frac{1}{8}$.
- Two smooth, identical spheres, P and Q, are resting on a smooth, horizontal surface. The masses of the spheres are 3m and m, respectively. P is projected towards Q with velocity u.
 - a Given that the coefficient of restitution between P and Q is e, show that, no matter the value of e, P will travel in the same direction as Q after the collision.
 - b Given that $e = \frac{2}{5}$, find the loss in kinetic energy due to the collision.
 - Three particles are lying in a straight line on a smooth, horizontal surface. The particles are labelled A, B and C, with B between A and C. The masses of the particles are 5 kg, 3 kg and 2 kg, respectively. The coefficient of restitution between A and B is $\frac{1}{2}$, and the collision between B and C is perfectly elastic. Particle C is projected towards particle B with velocity $8 \,\mathrm{m\,s^{-1}}$. Find the velocity of each particle after the second collision, stating whether or not there will be any further collisions.
 - The diagram shows a smooth sphere, travelling horizontally, that is colliding with two smooth vertical walls at a corner point. The sphere has an initial velocity u, and given that the coefficient of restitution between the sphere and wall is e, find a relationship between the initial velocity and the final velocity after two collisions.



- Particles P and Q are resting on a smooth, horizontal surface. A smooth, vertical wall is at a distance 2r from Q, and the wall is perpendicular to the line through P and Q. The masses of P and Q are 2m and m, respectively. The coefficient of restitution between P and Q is $\frac{2}{3}$, and the coefficient of restitution between Q and the wall is $\frac{2}{5}$. P is projected towards Q with velocity u. Find the distance of the particles from the wall when they collide for a second time.
- Three identical smooth spheres, A, B and C, of masses 2m, M and m, respectively, are resting on a smooth, horizontal surface. The coefficient of restitution between A and B is $\frac{4}{5}$, and the coefficient of restitution between B and C is $\frac{1}{2}$. Sphere A is projected towards B with velocity 2u. Given that A, B and C are collinear, and that the velocity of B is zero after it collides with C, find the loss in kinetic energy after the second collision.
- Two spheres, P and Q, which are of equal radii, are placed on a smooth, horizontal surface. Their masses are m and 2m, respectively. Initially, sphere Q is at rest and sphere P is projected with speed u towards Q. When the spheres collide, the direction of P is 30° to the line through the centres of P and Q. Given that the coefficient of restitution between the spheres is $\frac{4}{5}$, find the loss in kinetic energy due to the collision.



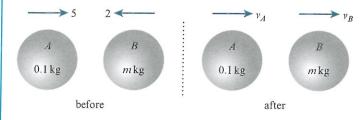
WORKED PAST PAPER QUESTION

Two smooth spheres A and B, of equal radii, have masses 0.1 kg and m kg respectively. They are moving towards each other in a straight line on a smooth horizontal table and collide directly. Immediately before the collision the speed of A is $5 \,\mathrm{m\,s^{-1}}$ and the speed of B is $2 \,\mathrm{m\,s^{-1}}$.

i Assume that in the collision A does not change direction. The speeds of A and B after the collision are $v_A \,\mathrm{m}\,\mathrm{s}^{-1}$ and $v_B \,\mathrm{m}\,\mathrm{s}^{-1}$ respectively. Express m in terms of v_A and v_B , and hence show that m < 0.25.

Cambridge International AS & A Level Further Mathematics 9231 Paper 2 Q4 November 2008

Answer



i Conservation of momentum: $0.1 \times 5 - m \times 2 = 0.1v_A + mv_B$, then $m = \frac{0.5 - 0.1v_A}{2 + v_B}$. Since $v_A > 0$ and $v_B > 0$, $0.5 - 0.1v_A < 0.5$ and $2 + v_B > 2$. So m < 0.25.

Checklist of learning and understanding

Momentum and impulse:

- Momentum is the product of mass and velocity such that P = mv.
- The rate of change of momentum is a force such that $F = \frac{dP}{dt}$.
- Impulse is given as I = m(v u), measured in newton seconds, Ns.
- Impulse is a force applied over time, so I = Ft, or Ft = mv mu.

Collisions:

- For the conservation of momentum, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$.
- For particles that coalesce (e = 0), $m_1u_1 + m_2u_2 = (m_1 + m_2)v$.
- For perfectly elastic collisions, e = 1.
- Newton's experimental law states that $e = \frac{\text{speed of separation}}{\text{speed of approach}}$, where e is known as the coefficient of restitution and can take values $0 \le e \le 1$.
- When two objects collide directly, conservation of momentum is considered along the line through their centres.
- When an object collides obliquely with a wall, only the component of the velocity that is perpendicular to the wall is considered for conservation of momentum.

END-OF-CHAPTER REVIEW EXERCISE 18

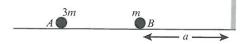


Two smooth spheres A and B, of equal radius, are moving in the same direction in the same straight line on a smooth horizontal table. Sphere A has mass m and speed u and sphere B has mass αm and speed $\frac{1}{4}u$. The spheres collide and A is brought to rest by the collision. Find the coefficient of restitution in terms of α . Deduce that $\alpha \ge 2$.

Cambridge International AS & A Level Further Mathematics 9231 Paper 2 Q3 November 2010



2



Two perfectly elastic small smooth spheres A and B have masses 3m and m respectively. They lie at rest on a smooth horizontal plane with B at a distance a from a smooth vertical barrier. The line of centres of the spheres is perpendicular to the barrier, and B is between A and the barrier (see diagram). Sphere A is projected towards sphere B with speed B and, after the collision between the spheres, B hits the barrier. The coefficient of restitution between B and the barrier is $\frac{1}{2}$. Find the speeds of A and B immediately after they first collide, and the distance from the barrier of the point where they collide for the second time.

Cambridge International AS & A Level Further Mathematics 9231 Paper 21 Q3 June 2010



Three small spheres, A, B and C, of masses m, km and 6m respectively, have the same radius. They are at rest on a smooth horizontal surface, in a straight line with B between A and C. The coefficient of restitution between A and B is $\frac{1}{2}$ and the coefficient of restitution between B and C is C. Sphere A is projected towards B with speed C and is brought to rest by the subsequent collision.

Show that k = 2.

Given that there are no further collisions after B has collided with C, show that $e \le \frac{1}{3}$.

Cambridge International AS & A Level Further Mathematics 9231 Paper 23 Q1 June 2011

CROSS-TOPIC REVIEW EXERCISE 3

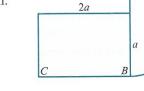
- A light, elastic string, of natural length 2m and modulus of elasticity 40 N, has one end attached to a ceiling at the point O. A particle of mass 0.5 kg is attached to the other end of the string, and the system is allowed to rest in equilibrium with the string taut.
 - a Find the extension in the string.

The particle is then pulled down a further 0.75 m and released.

- **b** Find the shortest distance between the particle and O in the subsequent motion.
- c Show that while the string is taut the particle performs simple harmonic motion.
- The diagram shows a lamina that consists of a uniform semicircular plate of radius a attached to a uniform rectangular plate of dimensions a and 2a.

The density of the rectangular section is twice the density of the semicircular section.

- a Find the centre of mass of the combined lamina from the edges AB and BC.
- **b** The lamina is suspended from the point A and allowed to hang in equilibrium. Find the angle between AB and the vertical.



- A particle of mass m is dropped from a great height. The air resistance on the particle is given as mkv N.
 - **a** Show that $\frac{\mathrm{d}v}{\mathrm{d}t} = g kv$.
 - **b** Hence, show that $v = \frac{g}{k}(1 e^{-kt})$.
 - c State the speed of the particle after a very long time.
- A smooth hemisphere, of radius 2a, is placed with its plane face on a horizontal surface. A particle, of mass 3m, is placed on the highest point of the hemisphere. The particle is then projected horizontally with speed $\sqrt{\frac{1}{4}ga}$.
 - a Find the height of the particle above the horizontal surface when it leaves the surface of the hemisphere.
 - **b** Find the speed of the particle when it strikes the horizontal surface.
- A particle is projected from the top of a cliff that is 25 m above the sea below. The speed of the projection is 40 m s⁻¹ and the angle of elevation is 15°. The particle travels as a projectile and lands in the sea.
 - a Find the horizontal distance the particle travels before landing in the sea.
 - **b** Find the duration of time for which the particle is 30 m above the sea below.
 - c Find the direction of the particle just before it hits the water.
- Two particles, P and Q, of masses 2m and 3m, respectively, are resting on a smooth, horizontal surface. Particle P is projected towards Q with speed u. It strikes Q directly and the coefficient of restitution between the particles is e.
 - a Find the velocity of P and Q after the collision, in terms of e and u.
 - **b** Find the range of values of e such that P's direction does not change due to the collision.
 - c Find the loss in kinetic energy due to the collision, giving your answer in terms of e, m and u.

FURTHER MECHANICS PRACTICE EXAM-STYLE PAPER

		The second secon	
1	t	A particle is projected from a point O , which is on horizontal ground: the speed of projection is $20 \mathrm{ms^{-1}}$ and he angle of elevation is 30° .	d
]	Find the speed and direction of the particle when it has travelled a horizontal distance of 30 m.	[5
2	F	A uniform, solid cylinder of radius $2a$ and length $7a$ has density ρ . The cylinder is attached to a hemisphere of radius $2a$ and density 2ρ . The plane face of the cylinder coincides with the plane face of the hemisphere.	
	C	This composite body is then suspended, in equilibrium, by a light string; this string is attached to a point on the rim of the plane face of the hemisphere. Find the angle which the plane face of the hemisphere make with the vertical.	es [7]
3	T1	A particle is travelling along a smooth, horizontal surface with speed $3u \mathrm{ms^{-1}}$. It has mass $2m$ and passes through the point O when $t=0$. An opposing force with magnitude $8t^2$ N is applied to the particle once it passes over the point O .	
	a	Find the velocity as a function of time.	[5]
	b	Find the time taken for the particle to pass through the point O again.	[4]
4	Δ		
	1.	A particle, of mass m , is resting in equilibrium on the inside surface of a smooth, circular hoop of radius $\frac{3}{2}a$. The hoop lies in a vertical plane. The particle is then projected with horizontal speed u from the lowest point on the inside of the hoop.	
	а	Show that, for complete circles, $u \ge \sqrt{\frac{15}{2}ga}$.	[6]
	b	Hence, find the range of values of u for which the particle does not remain in contact with the inner surface of the hoop.	[3]
5	OI	A particle, of mass 2 kg, is attached to one end of a light, elastic string of natural length 1.5 m and modulus of elasticity 50 N. The other end of the string is attached to a point on a ceiling. The particle is allowed to rest in equilibrium with the string taut.	
	a	The string is pulled down a distance of $0.1 \mathrm{m}$ to the point D and released. Show that, while the string is taut, the particle performs simple harmonic motion, stating the value of ω .	[6]
	Le	t the equilibrium position be denoted by the point E .	
	b	Find the time taken for the particle to travel from the point D to $0.08 \mathrm{m}$ above the point E .	[3]
6	Tw P is	o spheres, P and Q , of masses $2m$ and $5m$, respectively, are resting on a smooth horizontal surface. Sphere is projected with horizontal speed $3u$ and it subsequently strikes Q directly. The coefficient of restitution ween the spheres is $\frac{1}{2}$.	
	а	Find the speed of each particle after the collision.	[6]
	b	Find the loss in kinetic energy due to the collision.	[2]
	С	Find the magnitude of the impulse on P due to the collision.	[3]

[3]

2 H_0 : $\mu = 65$, H_1 : $\mu \neq 65$

Test statistic = -0.314

Critical value = $t_{0.95, 7} = -1.895$

There is insufficient evidence to suggest that the sprinklers are not activated at 65°C.

- 3 a Proof
 - **b** 0.614
- 4 i [520.0, 529.2] or 524.6 ± 4.6 [1]
 - ii H_0 : $\mu_b \mu_a = 0$, H_1 : $\mu_b \mu_a \neq 0$ $s^2 = 12.711$

Test statistic = 1.52

Critical value = 1.64

There is insufficient evidence to suggest a difference between the two means.

- 5 **a** $P(X = x) = p(1 p)^{x-1}$
 - **b** i $G_Y(t) = p^n t^n (1 qt)^{-n}$
 - ii $E(Y) = \frac{n}{p}$, $Var(Y) = \frac{nq}{p^2}$
- 6 H₀: No association between test results and school

H₁: An association between test results and school

Test statistic = 3.68

Critical value = $\chi_2^2(0.95) = 5.99$

Do not reject H₀: This is no association between test results and school.

7 H₀: There is no difference between the population medians

H₁: There is a difference between the population medians

$$R_m = 48 \text{ or } 88$$

$$W = 48$$

Critical value = 49

Reject H₀: There is a difference in the population medians, therefore there is a difference in the average height of trees on the two sides of the river.

- 8 a Proof
 - b Proof
 - c 1.74
- 9 H_0 : The population median is 147.50

H₁: The population median is greater than 147.50

$$P = 49, N = 6$$

$$T = \min(49, 6) = 6$$

Critical value = 10

Reject H_0 : There is sufficient evidence to suggest the median is greater than 147.50.

- 10 i $100 \times {}^{4}C_{2} \times 0.4^{2} \times 0.6^{2} = 34.56$
 - ii H_0 : A binomial B(4, 0.6) is a good model.

H₁: A binomial B(4, 0.6) is not a good model.

$$\chi^2 = 9.22$$

$$\chi_3^2(0.95) = 7.815$$

Reject H₀: Probability of faulty chips is not 0.6.

11 H_0 : Population median time taken = 140

 H_1 : Population median time taken > 140

Test statistic:
$$S^+ = 8(S^- = 2)$$

$$P(S^+ \ge 8) = 0.054688$$

Do not reject H₀: the time taken to fill in the forms is 140 minutes.

13 Projectiles

Prerequisite knowledge

- 1 magnitude of deceleration = $0.36 \,\mathrm{m\,s^{-2}}$
- 2 $t = 5.74 \,\mathrm{s}$ and $v = 37.4 \,\mathrm{m}\,\mathrm{s}^{-1}$

Exercise 13A

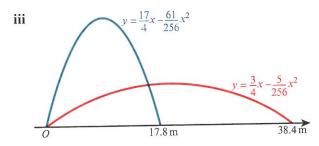
- 1 a $45.62 \,\mathrm{m \, s^{-1}}$
 - **b** 88.68 m
 - c 27.17° or 62.83°
- $2 42.75 \,\mathrm{m \, s^{-1}}$
- 3 $14.59 \,\mathrm{ms^{-1}} < u < 16.31 \,\mathrm{ms^{-1}}$
- 4 36.9°
- 5 $10\sqrt{2} < u < 20$
- 6 $7.75\,\mathrm{m\,s^{-1}}$
- 7 **a** $\sqrt{60} \,\mathrm{m \, s^{-1}}$
- **b** 2.75 m
- 8 $6\sqrt{15}$ m
- 9 2.62 s
- 10 a 2.32s
- **b** $25 \,\mathrm{m}\,\mathrm{s}^{-1}$

Exercise 13B

- 1 7.52 m
- 2 20 m
- 3 10.0 m
- 4 $39.9 \,\mathrm{m \, s^{-1}}$
- 5 30.9° below the horizontal
- **6 a** 0.77 s
- **b** $18.1 \,\mathrm{m \, s^{-1}}$
- 7 27.5 m
- 8 Proof
- 9 45°, 71.6°
- 10 Proof

End-of-chapter review exercise 13

- 1 i Proof
- ii 38.4 m, 17.8 m



- 2 i $y = x \frac{1}{10}x^2$
 - ii x = 4.23
 - iii $\theta = 8.8^{\circ}$
- 3 i $k = \frac{1}{2}$, 2, Proof ii t = 0.894 s
 - iii $16.3 \,\mathrm{m\,s^{-1}}$, 15.9 above the horizontal

14 Equilibrium of a rigid body

Prerequisite knowledge

- 1 Parallel component is $mg \sin \theta$, perpendicular is $mg \cos \theta$.
- **2** 30°

Exercise 14A

- 1 1.05 kg
- **2** 0.333 kg
- $3 \frac{10}{3}$

- 4 a 6.3 Nm anticlockwise
 - b 22.1 Nm anticlockwise
 - c 36 Nm anticlockwise
- 5 a 46.5 Nm clockwise
 - b 12.8 N m anticlockwise
 - c 2.89 N m anticlockwise
- 6 a 0.119 m
- **b** 3.8 kg
- $7 \qquad m = \frac{6}{19}M \frac{78}{19}$
- 8 **a** $x = 9.78 \,\mathrm{N}$
- **b** $x = 40.2 \,\mathrm{N}$
- $c x = 3.20 \,\mathrm{m}$

Exercise 14B

- 1 $\frac{179}{58}$ a from AB, $\frac{127}{58}$ a from AC
- 2 5.71 from AB, 2.64 from AC
- 3 1.28r
- 4 $G\left(\frac{197}{81}a, \frac{127}{81}a\right)$
- $\bar{x} = 3.04 \, r, \, \bar{y} = 1.91 \, r$
- $6 \quad \bar{x} = \frac{2}{2 + \pi}, \, \bar{y} = 1$
- $\overline{x} = 2.73 \, a, \, \overline{y} = 1.82 \, a$
- 8 Proof

Exercise 14C

- 1 2.34r
- 2 $\frac{5}{6}r$
- $\frac{3}{4}r$
- $4 \qquad \overline{x} = \frac{11}{8}a$
- $5 \qquad \overline{x} = \frac{99}{28}r$
- 6 $k = \frac{100}{3}$
- $7 \quad \bar{x} = 6.49 \, r$
- 8 $\frac{100a 9ka}{40 + 6k}$

Exercise 14D

- 68.2°
- 29.7°
- $\mu \geqslant \frac{1}{2\sqrt{3}}$ 3
- 32° 4
- 55.8°
- 6
- 0.474 mg, at an angle of 82.7° clockwise from the
- 8 Topples first
- a $\frac{11\sqrt{3}}{8}mg$

End-of-chapter review exercise 14

- i 562.5 N
- ii 52.1° with vertical
- 2 i 2.12 N
- ii $\mu = 0.313$
- 3 i 0.143 m
 - ii a P = 2.5
- **b** 0.787
- iii P = 26.0, Proof

15 Circular motion

Prerequisite knowledge

- Magnitude is 4.91 N, direction is 13.4° clockwise from the negative x-axis.
- $0.832\,\mathrm{m\,s^{-2}}$ 2
- 3 $21.0\,\mathrm{m\,s^{-1}}$

Exercise 15A

- **a** $\frac{2}{3}$ rad s⁻¹ **b** 2 m **c** 4 m s⁻¹

- $3.23 \, \text{rad s}^{-1}$
- $\omega = \sqrt{6} \, \text{rad s}^{-1}, 2.57 \, \text{s}$
- $\mu \geqslant \frac{1}{2}$
- $20 \, \text{m s}^{-1}$

- $7 \quad 2\sqrt{10} \,\mathrm{m\,s^{-1}}$

- $10 \quad \frac{3\sqrt{35}}{5} \leqslant v \leqslant \frac{\sqrt{435}}{5}$

Exercise 15B

- $2.17\,\mathrm{m\,s^{-1}}$
- $2.95\,\mathrm{rad}\,\mathrm{s}^{-1}$
- 3 $1.96\,\mathrm{rad}\,\mathrm{s}^{-1}$
- $27.65\,\mathrm{m\,s^{-1}}$ 4
- $T_{AP} = 37.2 \,\mathrm{N}, \, T_{BP} = 4.44 \,\mathrm{N}$
- **a** $\omega = 2.99 \, \text{rad s}^{-1}$
 - b 28.5 per minute
- $\mu = 0.305$
- a $4\sqrt{2}$ N
- **b** $6.51 \,\mathrm{m \, s^{-1}}$

Exercise 15C

- 48.2°
- 136.2°
- 44.0°
- $T_{\min} = 4mg$, $T_{\max} = 16mg$
- $u \leqslant \sqrt{2ga}$

- 8 **a** $\sqrt{15ga}$ **b** $v = \sqrt{3ga}$ **c** $R = \frac{1}{2}mg$ 9 **a** $v = \sqrt{\sqrt{2}ga}$ **b** $R_{\text{max}} = 3mg$ 10 **a** $\sqrt{7ga}$ **b** 18mg

 $\omega = 8.19 \,\mathrm{rad}\,\mathrm{s}^{-1}$, KE = 0.402 J

ii $T = 6.71 \text{ N}, \omega = 10.6 \text{ rad s}^{-1}$

16 Hooke's law

Prerequisite knowledge

21.6m

 $12.2\,\mathrm{m\,s^{-1}}$

Exercise 16A

a 0.45 m **b** 75 N **c** 1 N

0.125 m or 12.5 cm

15mg N 3

5mg N

1.93 m

6 $\lambda = 76.9 \,\text{N}$

7 **a** Proof **b** $7\sqrt{3}gN$

 $\mu = \frac{3}{4}, x = 1.2 \,\mathrm{m}$

1.22 m and 1.08 m

10 a $\frac{2ka}{2}$

b $k = \frac{3}{2}$

Exercise 16B

a 4.17 J **b** 2 m

c 250 N

Proof 2

 $\lambda = 62.5 \text{ N}, \text{ EPE} = 16 \text{ J}$ 3

 $\lambda = 6g, \frac{9}{10}gJ$

134.6J 5

a Proof

b k = 2

7 a 1.4 m

b 24.5 J

a $\frac{175}{16}a$

b $\frac{813}{32}$ mga J

Exercise 16C

24.6J 1

 $1.15\,\mathrm{m\,s^{-1}}$

a 1.07 m

b 2.07 m

4 $8.32 \,\mathrm{m \, s^{-1}}$

a $15.3 \,\mathrm{m \, s^{-2}}$ 5

b 38.5 J **c** 3.90 m

6 $4.68 \,\mathrm{m \, s^{-1}}$

7 a 1.13 m

h $4.19 \,\mathrm{m \, s^{-1}}$

 $EPE = \frac{5}{4}mgaJ, \frac{9}{8}mgaJ$

 $\sqrt{2g}\,\mathrm{m}\,\mathrm{s}^{-1}$

End-of-chapter review exercise 16

1 i Proof

ii $2.75\,\mathrm{m\,s^{-1}}$

i $T = 6.25 \,\mathrm{N}$, Proof ii $4.90 \,\mathrm{m \, s^{-1}}$

3 i $\lambda = 10 \,\mathrm{N}$

ii $1.12 \,\mathrm{m\,s^{-1}}$

17 Linear motion under a variable force

Prerequisite knowledge

1 **a** $12t^3$

 $\mathbf{b} \quad -\frac{1}{4}\cos 4t + c$

 $\mathbf{c} = \mathbf{e}^t \cos t - \mathbf{e}^t \sin t$

2 **a** $v = \sqrt{\frac{2}{3}t^3 + 2c}$ **b** $v = e^{2e^{-x} + c}$

Exercise 17A

1 **a** $v = 72 \,\mathrm{ms}^{-1}$, $a = 66 \,\mathrm{ms}^{-2}$

b $v = -3 \,\mathrm{m \, s^{-1}}, \ a = 4 \,\mathrm{m \, s^{-2}}$

 $v = 7.25 \,\mathrm{m \, s^{-1}}, a = 1.6875 \,\mathrm{m \, s^{-2}}$

 $2 8.89 \,\mathrm{m \, s^{-1}}$

4 $x = 3 - \frac{1}{t}$, cannot exceed 3 m

5 7.8 m s^{-1}

6 $v = 3 - \frac{1}{3}t^3, x = 3t - \frac{1}{12}t^4$

- 7 t=2
- No, takes 136.8 m to stop. 8
- 9.65 m 9
- Proof, $v_{\text{max}} = \frac{g}{l_s}$
- 11 a $\sqrt{6}$ s, or 2.45 s
- **b** 19.59 m
- 12 a Proof
- **b** $25 \,\mathrm{m}\,\mathrm{s}^{-1}$
- c 115 m

Exercise 17B

- $\mathbf{a} \quad v = \sqrt{\frac{2}{3}x^3 + 6x + k}$
 - **b** $v = \sqrt{e^{2x} x^2 + k}$
 - $\mathbf{c} \quad v = \sqrt{6x + \frac{2}{x} + k}$
- $6\sqrt{2}\,{\rm m}\,{\rm s}^{-1}$ 2
- $v = \sqrt{3x^2 + 25} \text{ m s}^{-1}$
- $0.0671 \,\mathrm{m\,s^{-1}}$
- $v = \sqrt{3e^{-2x} + 22}$, $\sqrt{22}$ m s⁻¹
- $4.19 \,\mathrm{m \, s^{-1}}$
- $v^2 = 6x + 2x^2 + 4$
- $a = \frac{40}{9\pi}$
- 9 **a** $v = \sqrt{32\sin\frac{x}{4} + 49}$ **b** $\sqrt{17} \,\mathrm{m} \,\mathrm{s}^{-1}$
- 10 $v = \sqrt{\frac{3}{2}x^2 + \frac{1}{x} + \frac{27}{2}}$

End-of-chapter review exercise 17

- i $x = 2.5 \,\mathrm{m}, v = 6.12 \,\mathrm{m \, s^{-1}}$ ii $a = -15 \,\mathrm{m \, s^{-2}}$
- i Proof
- ii Proof
- iii 49.7 m
- i $0.25v \frac{dv}{dx} = -(5-x)$ ii $x = 5(1-e^{-2t})$

18 Momentum

Prerequisite knowledge

- $v_x = 6\sqrt{3} \,\mathrm{m} \,\mathrm{s}^{-1}$ and $v_y = 6 \,\mathrm{m} \,\mathrm{s}^{-1}$
- 2 105J

Exercise 18A

- 1 $v_p = \frac{2}{3}u, v_q = \frac{8}{3}u$

- 5 $e = \frac{1}{4}$, KE lost $\frac{225}{16}$ J

- 7 **a** Proof **b** $e = \frac{1}{2}, v_p = 0$

Exercise 18B

- 0.901u
- $\frac{135}{32}$ mu² J

- 5 a Proof
- **b** $\frac{63}{200}mu^2$ J
- 6 $v_A = 3.6 \,\mathrm{m \, s^{-1}}, v_B = 0.4 \,\mathrm{m \, s^{-1}}, v_C = -1.6 \,\mathrm{m \, s^{-1}},$ No more collisions
- Final velocity is eu
- 9 $\frac{828}{625}mu^2$ J
- 10 $\frac{9}{100}$ mu² J

End-of-chapter review exercise 18

- 1 $e = \frac{4+\alpha}{3\alpha}$, Proof
- 2 $v_A = \frac{1}{2}u, v_B = \frac{3}{2}u, \frac{2}{5}a$
- 3 Proof

Cross-topic review exercise 3

- $1 \qquad e = \frac{2}{3}, \frac{5}{3}m\sqrt{2gh}$
- **2 a** 0.25 m **b** 1 m
- c Proof
- a $\frac{20a}{24+3\pi}$ from AB, $\left(\frac{4+\pi}{8+\pi}\right)a$ from BC
 - **b** 24.6°
- **a** Proof **b** Proof **c** $\frac{g}{k}$
- 5 **a** $\frac{17}{12}a$ **b** $2.06\sqrt{ga}$
- **6 a** 55.2 m
- **b** 0.536 s
 - c 32.5° above the horizontal
- 7 **a** $v_P = \frac{u}{5}(2 3e), v_Q = \frac{2u}{5}(1 + e)$

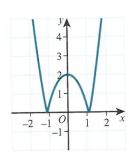
19 Hyperbolic functions

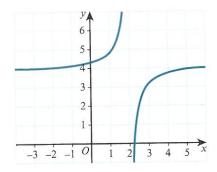
Prerequisite knowledge

- Proof
- $x = \ln 3$

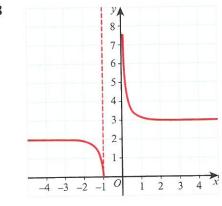
Exercise 19A

- $1 \qquad \ln\left(\frac{3}{5}\right)$
- $2 \quad \ln\left(\frac{4}{3}\right), \ln\left(\frac{5}{3}\right)$
- 3 $\log\left(\frac{1}{33}(8+\sqrt{97})\right)$
- 4 $\ln\left(\frac{2}{5}\right)$
- 5 $-\ln 2$ or $\ln \left(\frac{1}{2}\right)$





8



- $\ln 3$, $-\ln 5$
- 10 $\ln \frac{1}{2}$, $\ln \frac{4}{3}$
- 11 $\ln\left(\frac{3-\sqrt{5}}{2}\right)$, $\ln\left(\frac{3+\sqrt{5}}{2}\right)$

Exercise 19B

- Proof
- Proof
- 3 Proof
- 4 a Proof b Proof c Proof

Proof

Exercise 19C

- 1 **a** $\ln(3 + \sqrt{10})$ **b** $\frac{1}{2}\ln(5)$ **c** $\ln(2)$
- 2 **a** $\ln(\sqrt{10} 3), \ln(\frac{\sqrt{10} 1}{3})$
 - **b** 0
 - c $\ln\left(\frac{1+\sqrt{5}}{2}\right)$
- 3 Proof

Further Probability & Statistics practice exam-style paper

H₀: Recall for visual and aural is the same H₁: Recall from visual is better than aural.

Test statistic: $S^+ = 9(S^- = 3)$

 $P(S^+ \ge 9) = 0.072998$

Do not reject Ho:

Objects presented visually are not recalled more accurately than objects presented aurally.

2 a H₀: There is no difference between the population medians of the two departments

> H₁: There is a difference between the population medians of the two departments

$$m = 7, n = 8$$

$$R_m = 75$$

$$m(n+m+1) - R_m = 37$$

$$W = 37$$

Critical value = 38

Sufficient evidence to reject H₀: There is a difference in the population medians of the two departments.

b A normal approximation can be used with E(T) = 2525

Var(T) = 21042

 $T \approx N(2525, 21042)$

- **a** $E(X) = \frac{4}{3}, Var(X) = \frac{8}{9}$
 - **b** $G_Z(t) = \frac{1}{19683}(2+t)^9$
 - c E(Z) = 3, Var(Z) = 2
- H₀: No association between gender and facilities used

H₁: An association between gender and facilities

$$\sum \left(\frac{(O-E)^2}{E}\right) = 12.7$$

$$\chi_2^2(0.95) = 5.991$$

Reject Ho:

There is an association between gender and the facilities used at the hotel.

a 4.297

$$\mathbf{b} \quad \mathbf{F}(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{60}y(8+y) & 0 \le x < 2 \\ \frac{1}{96}(7x+18) & 2 \le x < 6 \\ \frac{1}{120}(30x-x^2-69) & x \ge 6 \\ 0 & \text{otherwise} \end{cases}$$

c Median = $\frac{30}{7}$

Further Mechanics practice exam-style paper

- Speed 18.8 m s⁻¹ and direction angle 22.9° below the horizontal.
- 2 49.3°
- **a** $v = 3u \frac{4}{3m}t^3$ **b** $t = (9mu)^{\frac{1}{3}}$
- a Proof b $\sqrt{3ga} < u < \sqrt{\frac{15}{2}ga}$
- 5 **a** $\ddot{x} = -\frac{50}{3}x$, $\omega = \sqrt{\frac{50}{3}}$
- 6 **a** $v_Q = \frac{9}{7}u$, $v_P = -\frac{3}{14}u$, opposite directions
 - **b** $\frac{135}{28}mu^2$ J **c** $\frac{45}{7}mu$ N

Further Pure Mathematics 2 practice exam-style paper

- 1 $y = \sec x \ln \sin x + c \sec x$
- 3 a $\begin{pmatrix} 1 & -2 & 1 & \vdots & 0 \\ 0 & 3 & 2 & \vdots & 2 \\ 0 & 0 & a-4 & \vdots & b-2 \end{pmatrix}$
 - **b** $a \neq 4, b \in R$
 - c a = 4, b = 2
 - **d** $a = 4, b \neq 2$
- 4 13
- Proof, $\ln(3 + \sqrt{8})$, $\ln(2 + \sqrt{3})$ 5

Glossary

 χ^2 -test: a test that can be used to look for the association between two sets of categorical data, or to perform a goodness of fit test

 ρ , the Greek letter rho: density (pronounced 'row' (spelt rho))

A

Acceptance region: the values of the test statistic for which we do not reject the null hypothesis

Angle of depression: the angle formed by the line of sight and the horizontal plane for an object below the horizontal

Angle of elevation: the angle formed by the line of sight and the horizontal plane for an object above the horizontal

Angular speed: the velocity of a body rotating about a fixed point, measured as the rate of change of the angle turned per unit of time

Arithmetic sequence: a sequence in which each successive term is obtained by adding the same constant value

Asymptote: a line that a curve tends towards

Augmented matrix: a matrix that is formed by combining the columns of two matrices

Auxiliary equation: an algebraic equation of degree n that is based upon an nth degree differential equation

B

Barycentre: the point between two objects, such as planets, where the objects are perfectly balanced with each other

Boundaries: limiting or bounding lines

Boundary conditions: a set of conditions that limit the possible solutions of differential equations

Breaking equilibrium: when the net force on an object is no longer zero

C

Cardioids: a type of polar curve that is heart shaped

Catenary: a naturally occuring shape observed in cases such as telephone cables and rope bridges, modelled by the hyperbolic cosine function

Cayley–Hamilton theorem: states that every square matrix satisfies its own characteristic equation

Centre of mass: the point at which the entire mass of a body may be considered

Characteristic equation: the polynomial of degree n that relates to the eigenvalues of a square matrix

Chi-squared: a family of distributions which are used to test association and goodness of fit

Circular functions: sine, cosine and tangent are called circular functions as they are derived from the unit circle

Circular motion: motion that occurs about a fixed point where the distance is constant

Coalesce: when two objects join together upon collision; occurs when there is zero elasticity between the objects

Coefficient of restitution: the measure of how elastic the collision is between two objects

Column: a vertical collection of terms, such as in a matrix **Common perpendicular:** when two or more lines or planes are such that a vector is at right angles to both of them

Comparison test: the use of a similar sum to compare against the original; this comparative sum is known to converge or diverge

Complementary function (CF): the general solution of the auxiliary equation of a linear differential equation

Composite: made up of several different parts or elements

Compressed: reduced in size due to external forces

Confidence interval: an interval for which there is a given probability that the population mean lies within that interval

Conic section: a special curve created by cutting through a right circular cone with a plane

Conical pendulum: a pendulum that performs horizontal circles about a centre that is vertically below where the string is attached

Conservation of energy: the total energy of an isolated system remains constant throughout the motion

Constraint: a rule or condition

Contingency table: a two-way table to display categorical data used in a chi-squared test

Convergent: a series is convergent if the sequence of its partial sums approaches a limit

Convolution theorem: in statistics, a theorem that allows evaluation of the probability generating function of the sum of two independent discrete random variables

Cross product: two vectors, \mathbf{u} and \mathbf{v} , are crossed to form a vector that is perpendicular to both \mathbf{u} and \mathbf{v}

Cube roots of unity: the three roots of the cubic equation $z^3 - 1 = 0$

Cubic equation: a polynomial with a leading term of power 3

Cumulative distribution function: a function that relates probability to the area under the graph for a probability density function that defines a continuous random variable

Cusp: a point on a curve at which two branches meet such that their tangents are equal

D

Deformation: the altering of the shape of an object

Degrees of freedom: the number of independent pieces of information that contribute to the estimate of a parameter

Denominator: the bottom portion of a fraction

Derivative: a function or value obtained from differentiating the original function

Determinant: a value obtained from the elements of a square matrix, usually used to represent the scaling factor from a transformation

Diagonalisable: a square matrix is known to be diagonalisable if it is similar to a diagonal matrix

Differential equation: an equation that contains the original function and at least the first derivative; the order of the differential equation is determined by the highest derivative in the equation

Differentiation: the process of finding the gradient function

Directly proportional: a relationship between two variables such that they increase in the same ratio

Discontinuity: a point on a curve in which f(a) does not exist; a gap exists in the curve

Discrete uniform distribution: a distribution in which the random variable takes specific values, and each value has an equal probability

Discriminant: a function obtained from the coefficients of a polynomial, allowing the deduction of the number of roots of the polynomial in question

Displacement: the position of an object relative to its starting point, measured as a vector

E

Eigenvalue: a value obtained from solving the characteristic equation of a square matrix

Eigenvector: a vector that maps to a factor of itself when a matrix is applied to it, the direction being unchanged

Elastic: a material that has the ability to stretch beyond its natural length when a force is applied to it

Elastic potential energy (EPE): the energy stored in an elastic body that has been stretched or compressed

Element: a value in a matrix

Ellipse: a curve surrounding two focal points, where the sum of the distances of a point on the curve to these two focal points is always constant

Energy: the measure of mechanical energy stored in a system, comprising kinetic energy, potential energy and elastic potential energy

Enlargement: a transformation that increases or decreases the area or volume of an existing shape, a stretch along all coordinate axes

Equilibrium: a state in which the resultant forces on an object are zero

Extension: the extra length created when an elastic object is stretched beyond its natural length

F

Free variable: a variable that does not correspond to a pivot column in a row reduced matrix

Frustum: a right circular cone with a smaller right circular cone cut off by slicing the cone to give a larger and smaller circular face

G

General solution (GS): a solution to a differential equation with undetermined constants

H

Homogeneous differential equation: a differential equation that includes terms in only one unknown function, e.g. y,

and its derivatives, e.g. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$. It is possible to arrange the terms to give zero on one side of the equation

Hooke's law: a law that relates the extension of a string or spring, or the compression of a spring, to the force applied

Hyperbola: a curve surrounding two focal points, where the difference of the distances of a point on the curve to these two focal points is always constant

Hyperbolic function: hyperbolic functions are derived from the unit hyperbola

Hyperbolic identities: relationships between hyperbolic functions similar to their trigonometric equivalents

1

Implicit: a function or expression that is not expressed directly in terms of independent variables

Impulse: a force applied over a given time interval

Induction: a method of proof in which a base case is shown to be true, then successive steps are shown also to be true, completing the proof

Inertia: the resistance of any physical object to change its current state of motion

Inextensible: a spring or string that cannot be stretched beyond its natural length

Inhomogeneous differential equation: a differential equation that can include terms in two different functions.

Moving all the terms in one function, e.g, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, to one

side of the equation leaves a function, e.g. f(x), on the other side instead of zero

Initial conditions: values that are defined or stated when the modelling of an observation is set in motion

Intersection: the point at which two or more objects, or functions, meet

Invariant: a point, or a set of points, that never change their value

Inverse matrix: a square matrix that can be multiplied by the original matrix to produce an identity matrix

Iteration: a repeat of a mathematical procedure applied to the result of a previous iteration

L

Lamina: a 2-dimensional surface with both mass and density

Limiting friction: a maximum value of static friction for which motion is impeded

Line of intersection: a line that is common to two or more planes in 3-dimensional space

Linear motion: motion that occurs in a straight line; it can be described with one spatial dimension

Logarithmic form: meaning that the answer should be written in exact form, using logarithms, usually ln

M

Matrix (plural: matrices): a rectangular array that consists of elements that are numbers or expressions, arranged in rows and columns

Model: an equation or system of equations that are used to closely resemble an observed phenomenon

Modulus of elasticity: a value that measures the resistance of an object to being stretched or compressed

Moment: a turning effect produced by a force acting at a distance on an object

Momentum: the quantity of motion of a moving body, measured as a product of its mass and velocity

N

Natural length: the original length of a spring or string before any forces act upon it

Newton's equations of motion: the set of equations that govern motion where the acceleration is constant

Newton's experimental law: a law that relates the velocities of two objects before and after collision

Non-parametric test: a test that does not require knowledge of the underlying distribution

Non-singular matrix: a matrix that has a non-zero determinant and an inverse

*n*th roots of unity: the *n* solutions of the complex equation $z^n = 1$

0

Oblique: a type of asymptote that is neither horizontal nor vertical

Oblique collision: a collision in which the line of centres of the two objects is not parallel to both the objects' direction of motion

Order (of a matrix): the order of a matrix is the size of the matrix defined by the number of rows (m) and columns (n) and written $m \times n$

Osborne's rule: a rule that changes trigonometrical identities to hyperbolic identities

P

Parabola: a plane curve formed by intersecting a right circular cone with a plane that is parallel to the generator of the cone

Parabolic trajectory: a trajectory modelled by motion in a 2-dimensional plane for which the acceleration is constant

Particle: a small point mass used to represent a larger object

Particular integral (PI): a function used to convert inhomogeneous equations to homogeneous equations

Particular solution: a solution generated by given initial or boundary conditions

Percentile: a measure indicating the value below which a given percentage of observations in a group of observations fall

Perfectly elastic: a collision between two particles in which no kinetic energy is lost

Piecewise function: a function which is defined by several sub-functions, each applying to a certain interval of the domain

Polar coordinates: a 2- or 3-dimensional system for which the distance from the origin and the angle turned through are ordinates

Polynomial: a function consisting of many terms of a variable, with each term having a different non-negative integer power

Position vector: a vector that measures displacement from a given origin

Primitive: the inverse of a derivative, an indefinite integral **Probability density function:** a function that describes the relative likelihood for the random variable to take on a given value

Probability generating function: a function that describes the probability of the discrete variable having a value, but in the form of a polynomial

Projectile: a particle or object that, once thrown, continues to move under its own inertia and the force of gravity

O

Quadratic: a polynomial with a leading term of power 2 Quartic: a polynomial with a leading term of power 4

R

Rational function: an algebraic fraction in which the numerator and the denominator are polynomials Reduced row echelon form: a matrix that has only the leading diagonal of elements that are non-zero

Reduction: a way of simplifying an integral through integration by parts and a recurrence relation

Reflection: a transformation in which all points in the image are equidistant from a mirror line with their original positions

Rigid body: a body that remains in equilbrium in all directions

Root: a solution of an equation

Rotation: a transformation in which a plane figure rotates about a fixed point

Row: a horizontal collection of terms, such as in a matrix Row echelon form: a matrix that has a lower triangle of zeros and a leading diagonal of non-zero elements

S

Scalar equation of a plane: the standard definition of a plane, written in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

Scalar product: the result of projecting the length of one vector parallel to the direction of another vector

Sequence: a set of mathematically ordered values or terms

Series: the sum of the terms of a sequence

Shearing: each point in a shape is displaced by an amount that is proportional to its distance from a fixed parallel invariant line

Singular matrix: a matrix that has a zero determinant, and as a consequence it cannot be inverted

Stretch: a type of transformation in which curve, or shape has either its *x* or *y* values changed by a scale factor

T

Top-heavy fraction: a fraction where the numerator's algebraic expression is the same degree or higher than that of the denominator's expression

Turning point: a point on a curve at which the gradient is equal to zero and where the gradient is of a different sign on either side of the turning point

H

Uniform: identical or consistent throughout **Unit hyperbola:** a curve with the equation $x^2 - y^2 = 1$ **Unit vector:** a vector of magnitude 1

V

Vector determinant: see Determinant

Vector equation of a line: the vector representation of a line, written in the form $\mathbf{r} = \mathbf{a} + \mathbf{b}t$

Vector equation of a plane: the vector representation of a plane, written in the form $\mathbf{r} = \mathbf{a} + \mathbf{b}s + \mathbf{c}t$

Vector product: the crossing of two vectors to create a common perpendicular vector, also known as the cross product

Z

Zero matrix: a matrix that has all its elements as zeros